Shock Waves

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1 Introduction

A shock wave is a sudden or even violent change in pressure occurring in a thin layer-like region of a continuous medium such as air. Other properties of the medium also change suddenly: notably the velocity, temperature, and entropy. In a strong shock, produced for example by a spacecraft while still in the earth's atmosphere, the increase in temperature is great enough to break up the molecules of the air and produce ionization. Shock waves that are "weak" in the mathematical sense (i.e., those that produce a pressure change that is only a minute fraction of atmospheric pressure) are, unfortunately, extremely strong in the subjective sense, because of the exquisite sensitivity of our hearing. This fact is of great importance for civil aviation in limiting the possible routes of supersonic aircraft, which produce shock waves with focusing properties and locations that depend sensitively on atmospheric conditions.

In the modeling of a shock wave, the thin region of rapid variation may nearly always be replaced by a surface across which the properties of the medium are regarded as discontinuous. The governing equations of motion, representing conservation of mass, momentum, and energy, still apply if they are expressed in integrated form, and their application to a region containing the surface of discontinuity leads to simultaneous algebraic equations relating the limiting values of physical quantities on opposite sides of this surface. In addition, the entropy of the medium must increase on passage through the shock. The resulting algebraic relations, in conjunction with the partial differential equations of motion applied everywhere except on the shock surface, suffice for the solution of many important practical problems. For example, they determine the location and speed of propagation of the shock wave, which are usually not given in advance but have to be determined mathematically as part of the process of solving a problem.

The theory of shock waves has important military applications, especially to the design of high-speed missiles and the properties of blast waves produced by explosions. On occasion, this has stimulated world-class mathematicians (and even philosophers) to make contributions to the subject that have turned out to be enduringly practical. Ernst Mach, who with Peter Salcher in the 1880s was the first person to photograph a shock wave, explained the properties of the high-speed bullets that had been used in the Franco-Prussian war (1870–71), and in World War II John von Neumann analyzed the effect of blast waves on tanks and buildings, obtaining the surprising result that a shock wave striking obliquely exerts a greater pressure than if it strikes head-on. Richard von Mises and Lev Landau also worked on shock waves in this period, as well as fluid and solid mechanics "full-timers" such as Geoffrey I. Taylor and Theodore von Karman.

2 Mathematical Theory

2.1 Mass, Momentum, and Energy

For definiteness, consider a shock wave in air. First we present the equations that apply to a *normal shock*, through which the air flows at right angles. Later we consider an *oblique shock*, for which arbitrary orientations (within wide limits) are possible between the incoming and outgoing air, and the shock itself.

Three jump conditions across a normal shock are

$$\rho_1 u_1 = \rho_2 u_2,$$
$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2,$$
$$e_1 + \frac{p_1}{\rho_1} + \tfrac{1}{2}u_1^2 = e_2 + \frac{p_2}{\rho_2} + \tfrac{1}{2}u_2^2.$$

The subscripts 1 and 2 here refer to opposite sides of the shock, and the frame of reference is that in which the shock is at rest. The variables are the densities ρ_1, ρ_2, the velocity components u_1, u_2, the pressures p_1, p_2, and the internal energies per unit mass e_1, e_2. All quantities may be taken to be positive, with the air flowing in from side 1 and out from side 2.

The first of the above equations represents conservation of mass, since the left-hand side is the mass of air per unit area per unit time flowing *into* the shock, and the right-hand side is the corresponding quantity flowing *out*. The second equation accounts for the change in the momentum of air per unit volume from $\rho_1 u_1$ to $\rho_2 u_2$. This change occurs at a rate proportional to the velocity, giving the terms $\rho_1 u_1^2$, $\rho_2 u_2^2$, and is driven by the net force arising from the jump in pressure, giving the terms p_1, p_2. The third equation accounts for the change in the energy of the air per unit mass from $e_1 + \frac{1}{2}u_1^2$ to $e_2 + \frac{1}{2}u_2^2$, comprising internal energy and kinetic energy. The forces that produce this change in energy come from the pressure, and give the terms

p_1/ρ_1, p_2/ρ_2. It is convenient to combine each internal energy and pressure term into a single quantity: the enthalpy. Then, in terms of the enthalpies h_1, h_2, defined by $h_1 = e_1 + p_1/\rho_1$ and $h_2 = e_2 + p_2/\rho_2$, the energy equation is

$$h_1 + \tfrac{1}{2}u_1^2 = h_2 + \tfrac{1}{2}u_2^2.$$

2.2 Entropy

A fourth jump condition across a shock concerns the entropy, which is a measure of the energy in the disordered molecular motions that is not available to perform work at the macroscopic scale. The condition is that the entropy of the material passing through the shock must increase. Although the entropy condition is "only" an inequality, it is of fundamental importance in determining the type of shocks that can or cannot occur. For example, the condition shows that, in almost all materials, a rarefaction shock (at which the pressure suddenly falls) is impossible. A shock in air therefore has the property that on passing through it the air undergoes an increase in pressure, density, temperature, internal energy, and enthalpy, and a decrease in velocity.

For air in conditions that are not too extreme, the entropy per unit mass per degree on the absolute temperature scale is proportional to $\ln(p/\rho^\gamma)$, where γ is the ratio of specific heats, which is approximately a constant. Thus, with the sign convention that u_2 and u_1 in the jump conditions are positive, the entropies s_2 and s_1 on opposite sides of the shock satisfy the condition $s_2 > s_1$, so that

$$p_2/\rho_2^\gamma > p_1/\rho_1^\gamma.$$

2.3 The Rankine–Hugoniot Relation

Simple algebraic manipulation of the jump conditions gives

$$h_2 - h_1 = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(p_2 - p_1).$$

This is an especially useful relation because h_2 and h_1 are functions of (p_2, ρ_2) and (p_1, ρ_1), respectively. Thus, if (p_1, ρ_1) are fixed, representing known upstream conditions, the Rankine–Hugoniot relation gives the *shock adiabatic*, relating p_2 to ρ_2, i.e., the pressure and density downstream.

2.4 The Mach Number

A basic property of a normal shock, deducible from the four jump conditions, is that the flow into it is supersonic, i.e., faster than the speed of sound, and the flow out of it is subsonic, i.e., slower than the speed of sound. This provides a strong hint that we should use variables based on the incoming and outgoing *Mach numbers*, M_1 and M_2, defined as the ratio of the flow speeds to the local speed of sound, denoted by c_1 and c_2. Thus

$$M_1 = \frac{u_1}{c_1} > 1, \qquad M_2 = \frac{u_2}{c_2} < 1.$$

We also have available the equation of state for air, $p = \rho RT$, and the innumerable formulas of thermodynamics, from which we select merely

$$c^2 = \gamma RT = \gamma p/\rho = (\gamma - 1)h.$$

Here R is the gas constant, T is absolute temperature, and the formulas apply with subscript 1 or subscript 2 (but the same values of γ and R) on each side of the shock. We now have simultaneous equations in abundance, and a little algebra starting from the jump conditions gives

$$M_2^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{\gamma M_1^2 - \frac{1}{2}(\gamma - 1)}.$$

2.5 Pressure and Temperature Rise

With the aid of the relation between M_2^2 and M_1^2, the change in any quantity at a normal shock may readily be expressed in terms of M_1^2 and γ alone. Such a representation is highly practical in calculations. For example, an important aspect of a shock wave is that it can produce considerable increases in pressure and temperature. But how do these increases depend on the Mach number of the incoming flow? The answer for a shock wave in air is

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)$$

and

$$\frac{T_2}{T_1} = \frac{(1 + \frac{1}{2}(\gamma - 1)M_1^2)(\gamma M_1^2 - \frac{1}{2}(\gamma - 1))}{\frac{1}{4}(\gamma + 1)^2 M_1^2}.$$

The functional form of this last equation is important for modeling purposes. Because the specific-heat ratio γ is of "order one" (it is approximately 1.4 for atmospheric air), the equation shows that at high Mach numbers the absolute temperature ratio T_2/T_1 scales with M_1^2. Thus, during the acceleration phase of the flight of a hypersonic vehicle, the simple equation of state we have been using, $p = \rho RT$, ceases to apply long before operating speed is attained. Physical processes beyond the reach of this "ideal gas equation" must be included, and the subject of shock waves rapidly starts to overlap with the chemistry of the breakup of the molecules, and with the physics of ionized gases. Applied mathematicians such as Sir James Lighthill and John F. Clarke have played an important role in developing the required theory of such strong shock waves.

3 Oblique Shock Waves

3.1 Usefulness in Modeling

Oblique shock waves provide a versatile tool for modeling the flow around a supersonically moving body of any shape. The reason for this is that the local change in flow direction produced by the surface of the body always produces an oblique shock somewhere in the flow. A concave corner, for example, produces an oblique shock attached to the corner, and the apex of a forward-facing wedge produces oblique shocks on each side of the wedge, attached to the apex.

Moreover, a smooth concave part of the body surface produces a flow field in which disturbances propagate on surfaces whose orientation is that of oblique shocks of vanishingly small strength. These *Mach surfaces* come to a focus a short distance away from a concave boundary, to produce a full-strength oblique shock. In general, a Mach surface is a surface on which a weak disturbance in pressure and associated quantities can propagate in accordance with the equations of acoustics as a *Mach wave*. The theory of weak shock waves is subsumed under acoustic theory in such a way that a weak shock corresponds to a characteristic surface of the acoustic wave equation. Shocks and characteristic surfaces do not in general coincide, but they do in the limit of zero shock strength.

Oblique shock waves are fundamental to the modeling of shock wave intersections and reflections. In a fluid, shock intersections produce vorticity, localized in surfaces called *slip surfaces*, across which there is a jump in the tangential component of velocity. Shock reflections can be of different types, depending on the orientation and strength of the incoming shock. A common type is the *Mach reflection*, in which a nearly straight shock wave, known as the stem, extends from the body surface to a triple intersection of shocks, from which there also emerges a slip surface.

3.2 Flow Deflection

The most basic question one can ask about an oblique shock is the following: if the incoming flow is at Mach number M_1, and the flow is deflected by an angle θ from its original direction, what are the possible oblique shock angles ϕ that can bring this deflection about? The angle ϕ is that between the shock surface and the incoming flow, so that a normal shock corresponds to $\theta = 0$ and $\phi = \pi/2$. The question is readily answered by starting with the formulas for a normal shock and superposing a uniform flow parallel to the shock surface. A little algebra and trigonometry give

$$\tan\theta = \frac{(M_1^2 \sin^2\phi - 1)\cot\phi}{1 + (\frac{1}{2}(\gamma+1) - \sin^2\phi)M_1^2}.$$

At a single value of M_1, the graph of θ against ϕ rises from $\theta = 0$ at $\phi = \sin^{-1}(1/M_1)$ to a maximum $\theta = \theta_{\max}(M_1)$ at a higher value of ϕ, before falling back to $\theta = 0$ at $\phi = \pi/2$. Thus the shock angle and flow deflection are limited to definite intervals. The angle $\sin^{-1}(1/M_1)$ is the *Mach angle*, and gives the Mach wave described above.

As M_1 is varied, there is a maximum possible value of $\theta_{\max}(M_1)$, attained in the limit of large M_1, and there is a corresponding value of ϕ. For air in conditions that are not too extreme, with a ratio of specific heats close to $\gamma = 1.4$, we obtain the useful result that the greatest possible flow deflection at an oblique shock is 46°, and the corresponding shock angle is 68°.

For given M_1, and a deflection angle in the allowable range, there are two possible shock angles ϕ. The smaller angle gives the weaker shock, and, except in a narrow range of deflection angles just below $\theta_{\max}(M_1)$, a supersonic outflow; the outflow from the stronger shock, at the larger angle, is always subsonic. Thus, although the inflow must be supersonic, the outflow may be subsonic or supersonic, depending on the magnitude of the equivalent superposed uniform flow on a normal shock referred to above. The ultimate explanation of such matters is the entropy condition. Whether or not the outflow is supersonic is important in practice, because a supersonic outflow may contain further shock waves. Which shock pattern actually occurs may depend on the downstream conditions.

4 Dramatic Examples of Shock Waves

4.1 Asteroid Impact

The earth has a long memory: the shock wave produced by the asteroid that extinguished the dinosaurs (and hence led to human life, via the rise of mammals) left a signature that is still visible in the earth's crust. The asteroid struck earth 66 million years ago near

Chicxulub in the Yucátan Peninsula in Mexico, producing a crater 180 km in diameter. Besides generating a tsunami in the oceans, which mathematically speaking is a type of shock wave, it generated shock waves in the rocks of the earth's crust. These shock waves propagated around the earth, triggering an abundance of violent events, such as earthquakes and the eruption of volcanoes.

Geologists have determined which metamorphic features of rock can only be produced by the intense pressure in the shock waves from a meteorite or asteroid, one example being the patterns of closely spaced parallel planes that are visible under a microscope in grains of shocked quartz. The mathematical theory of shock waves in solids, such as metal and rock, is highly developed.

4.2 The Trinity Atomic Bomb Test

The world's first nuclear explosion took place in July 1945, in the desert of New Mexico in the United States, when the U.S. army tested an atomic bomb with the code name Trinity. The explosion was filmed, and the film, which showed the hemispherical shock wave rising into the atmosphere, was released not long afterward.

For all the complexity of the physical processes and fluid dynamics in the explosion, it is remarkable that a simple scaling law describes to high accuracy the radius r of the shock wave as a function of time t after detonation, in which the only parameters are the explosion energy E and the initial air density ρ. The scaling law is

$$r = C\left(\frac{Et^2}{\rho}\right)^{1/5},$$

where C is a constant close in value to 1. With the aid of this scaling law, and a logarithmic plot, G. I. Taylor was able to determine the energy of the explosion rather accurately using only the film as data.

4.3 Shock Wave Lithotripsy

A medical treatment for a kidney stone is to break up the stone by applying a pulsed sound wave of high intensity that is brought to a focus at the stone. The focusing causes the pulses to become shock waves, in which the sudden rise in pressure provides the required disintegrative force on the stone. The machine used is called a lithotripter; it is designed so that the frequency and intensity of the shock waves can be controlled and varied as the treatment progresses. A water bath is applied to the patient's back, so that the pulse propagates through water and then tissue.

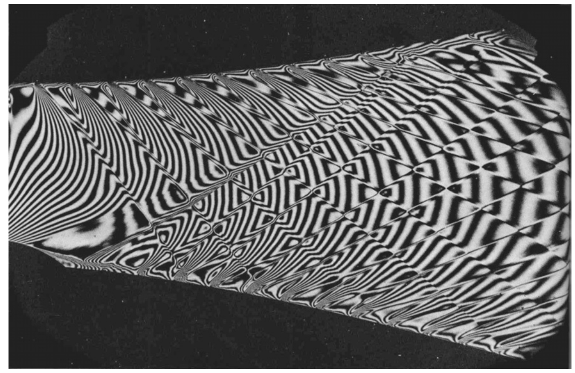

Figure 1 Weak shock waves (Mach surfaces) in a supersonic nozzle.

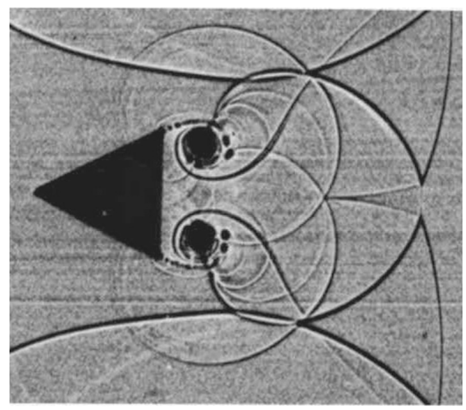

Figure 2 Reflection of a shock wave by a wedge, and rolling up of the vortex sheets (slip surfaces) produced.

5 Further Reading

Many excellent photographs of oblique shock wave patterns may be found in Van Dyke (1982). Figures 1 and 2 are examples of science as art.

Chapman, C. J. 2000. *High Speed Flow*. Cambridge: Cambridge University Press.

Ockendon, H., and J. R. Ockendon. 2004. *Waves and Compressible Flow*. New York: Springer.

Van Dyke, M. 1982. *An Album of Fluid Motion*. Stanford, CA: Parabolic Press.