Aircraft Noise

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1 Introduction

Aircraft noise is a subject of intense public interest, and it is rarely out of the news for long. It became especially important early in the 1950s, with the growth in the use of jet engines, and again in the 1960s with the development of the Anglo-French supersonic aircraft Concorde (now retired). Aircraft noise is currently a major part of the public debate about where to build new airports, and whether they should be built at all.

The general public might be surprised to know that our understanding of the principles of aircraft noise, and of the main methods that are available to control it, have come largely from mathematicians. In particular, Sir James Lighthill, one of the great mathematical scientists of the twentieth century, created a new scientific discipline, called aerodynamic sound generation, or aeroacoustics, when he published the first account of how a jet generates sound in 1952. Prior to this theory, which he created by mathematical analysis of the equations of fluid motion, no means were available to estimate the acoustic power from a jet even to within a factor of a million.

Moreover, Lighthill's theory had an immediate practical consequence. The resulting eighth-power scaling law for sound generation as a function of Mach number (flow speed divided by sound speed) meant that the amount of thrust taken from the jet would have to be strictly controlled; this was achieved by the development of high bypass-ratio turbofan aeroengines, in which a high proportion of the thrust is generated by the large fan at the front of the engine. Such a fan produces a relatively small increase in the velocity of a large volume of air, i.e., in the bypass air flow surrounding the jet, in contrast to the jet itself, which produces a large increase in the velocity of a small volume of air. The engineering consequence of a mathematical theory is therefore plain to see whenever a passenger ascends the steps to board a large aircraft and admires the elegant multibladed fan so clearly on display.

2 Lighthill's Acoustic Analogy

The idea behind Lighthill's theory is that the equations of fluid dynamics may be written exactly like the wave equation of acoustics, with certain terms (invariably written on the right-hand side) regarded as acoustic sources. Solution of this equation gives the sound field generated by the fluid flow. A key feature of the method is that the source terms may be estimated using the simpler aerodynamic theory of flow in which acoustic waves have been filtered out. The simpler theory, which includes the theory of incompressible flow as a limiting case, had become very highly developed during the course of the two world wars and in the period thereafter, and was therefore available for immediate use in Lighthill's theory of sound generation.

This innocuous-sounding description of the method hides many subtleties, which are in no way alleviated by the fact that Lighthill's equation is exact. But let us first give the equation in its standard form and define its terms. The equation is

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right)\rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

Here, t is time and $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ is the Laplacian operator, where (x_1, x_2, x_3) is the position vector in Cartesian coordinates. Away from the source region, the fluid (taken to be air) is assumed to have uniform properties, notably sound speed c_0 and density ρ_0. At arbitrary position and time, the fluid density is ρ, and the deviation from the uniform surrounding value is $\rho' = \rho - \rho_0$, the density perturbation. The left-hand side of Lighthill's equation is therefore simply the wave operator, for a uniform sound speed c_0, acting on the density perturbation produced throughout the fluid by a localized jet.

The right-hand side of the equation employs the summation convention for indices (here over $i, j = 1, 2, 3$), and so is the sum of nine terms in the second derivatives of $T_{11}, T_{12}, \dots, T_{33}$. Collectively, these nine quantities T_{ij} form the Lighthill stress tensor, defined by

$$T_{ij} = \rho u_i u_j + (p' - c_0^2\rho')\delta_{ij} - \tau_{ij}.$$

Here, (u_1, u_2, u_3) is the velocity vector, $p' = p - p_0$ is the pressure perturbation, defined analogously to ρ', and τ_{ij} is the viscous stress tensor, representing the forces due to fluid friction. Because of the symmetry of T_{ij} with respect to i and j, there are only six independent components, of which the three with $i = j$ are longitudinal and the three with $i \neq j$ are lateral.

The solution of Lighthill's equation, together with the acoustic relation $p' = c_0^2\rho'$ that applies in the radiated sound field (though not in the source region), gives the outgoing sound field $p'(\boldsymbol{x}, t)$ in the form

$$p' = \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{T_{ij}(\boldsymbol{y}, t - |\boldsymbol{x} - \boldsymbol{y}|/c_0)}{|\boldsymbol{x} - \boldsymbol{y}|}\,\mathrm{d}V.$$

Here, $\boldsymbol{x}$ is the observer position, $\boldsymbol{y} = (y_1, y_2, y_3)$ is an arbitrary source position, and $|\boldsymbol{x} - \boldsymbol{y}|$ is the distance between them. The integration is over the source region V, and the volume element is $\mathrm{d}V = \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3$. The retarded time $t - |\boldsymbol{x} - \boldsymbol{y}|/c_0$ allows for the time it takes a signal traveling at speed c_0 to traverse the distance from $\boldsymbol{y}$ to $\boldsymbol{x}$.

Lighthill's equation is referred to as an acoustic analogy, because the uniform "acoustic fluid," in which acoustic perturbations are assumed to be propagating, is entirely fictional in the source region! For example, the speed of sound there is not uniform, but varies strongly with position in the steep temperature gradients of an aircraft jet; and the fluid velocity itself, which contributes to the propagation velocity of any physical quantity such as a density perturbation, is conspicuously absent from a wave operator that contains only c_0. Nevertheless, Lighthill's power as a mathematical modeler enabled him to see that the acoustic analogy, with T_{ij} estimated from nonacoustic aerodynamic theory, would provide accurate predictions of jet noise in many important operating conditions, e.g., in subsonic jets at Mach numbers that are not too high.

The increasing difficulty in applying Lighthill's theory at higher Mach numbers derives from the fact that an aircraft jet is turbulent, and contains swirling eddies of all sizes that move irregularly and unpredictably throughout the flow. The higher the Mach number, the more that must be known about the individual eddies or their statistical properties in order to take account of the delicate cancelation of the sound produced by neighboring eddies. As the Mach number increases, small differences in retarded times become increasingly important in determining the precise amount of this cancelation. Lighthill's equation therefore provides an impetus to develop ever more sophisticated theories of fluid turbulence for modeling the stress tensor T_{ij} to greater accuracy.

2.1 Quadrupoles

A noteworthy feature of Lighthill's equation is that the source terms occur as second spatial derivatives of T_{ij}. This implies that the sound field produced by an individual source has a four-lobed cloverleaf pattern, referred to as a quadrupole directivity. A crucial aspect of Lighthill's theory of aerodynamic sound generation is, therefore, that the turbulent fluid motion creating the sound field is regarded as a continuous superposition of quadrupole sources of sound. Physically, this is the correct point of view, because away from boundaries the fluid has "nothing to push against except itself"; that is, since the pressure inside a fluid produces equal and opposite forces on neighboring elements of fluid, the total dipole strength, arising from the sum of the acoustic effects of these internal forces, must be identically zero.

Mathematically, the four-lobed directivity pattern arises from the expansion of p' in powers of $1/|\boldsymbol{x}|$ for large $|\boldsymbol{x}|$. The dominant term, proportional to $1/|\boldsymbol{x}|$, gives the radiated sound field; its coefficient contains angular factors $x_i x_j/|\boldsymbol{x}|^2$ corresponding to the derivatives $\partial^2/\partial x_i \partial x_j$ acting on the integral containing T_{ij}. For example, in spherical polar coordinates (r, θ, ϕ) with $r = |\boldsymbol{x}|$ and

$$(x_1, x_2, x_3) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta),$$

the lateral (x_1, x_2) quadrupole has the directivity pattern

$$\frac{x_1 x_2}{|\boldsymbol{x}|^2} = \tfrac{1}{2} \sin^2\theta \sin 2\phi.$$

This pattern has lobes centered on the meridional half-planes $\phi = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$, separated by the half-planes $\phi = 0$, $\pi/2$, π, and $3\pi/2$ in which there is no sound radiation.

2.2 The Eighth-Power Law

Lighthill's theory predicts that the power generated by a subsonic jet is proportional to the eighth power of the Mach number. This follows from a beautifully simple scaling argument based on only the equations given above, and we will now present this argument in full, at the same time giving the scaling laws for the important physical quantities in the subsonic jet noise problem.

Assume that a jet emerges with speed U from a nozzle of diameter a into a fluid in which the ambient density is ρ_0 and the speed of sound is c_0. The Mach number of the jet is therefore $M = U/c_0$, and the timescale of turbulent fluctuations in the jet is a/U. The frequency therefore scales with U/a, and the wavelength λ of the radiated sound scales with $c_0 a/U$, i.e., a/M. We are considering the subsonic regime $M < 1$, so $a < \lambda$, and the source region is compact.

The Lighthill stress tensor T_{ij} is dominated by the terms $\rho u_i u_j$, which scale with ρU^2, and the integration over the source region gives a factor a^3. In the far field, the term $|\boldsymbol{x} - \boldsymbol{y}|$ appearing in the denominator of the integrand may be approximated by $|\boldsymbol{x}|$, and the spatial derivatives applied to the integral introduce a factor $1/\lambda^2$. The far-field acoustic quantities are the perturbation pressure, density, and velocity, i.e., p', ρ', and u', which satisfy the acoustic relations $p' = c_0^2\rho' = \rho_0 c_0 u'$. Putting all this together gives the basic far-field scaling law

$$\frac{p'}{\rho_0 c_0^2} = \frac{\rho'}{\rho_0} = \frac{u'}{c_0} \sim M^4 \frac{a}{|\boldsymbol{x}|}.$$

In a sound wave, the rate of energy flow, measured per unit area per unit time, is proportional to $p'u'$. Since a sphere of radius $|\boldsymbol{x}|$ has area proportional to $|\boldsymbol{x}|^2$, it follows that the acoustic power W' of the jet, i.e., the total acoustic energy that it radiates per unit time in all directions, obeys the scaling law

$$\frac{W'}{\rho_0 c_0^3 a^2} \sim M^8.$$

This is the famous eighth-power law for subsonic jet noise. It would be hard to find a more striking example of the elegance and usefulness of the best applied mathematics.

The significance of the high exponent in the scaling law is that at very low Mach numbers a jet is inefficient at producing sound; but this inefficiency does not last when the Mach number is increased. This is why jet noise became such a problem in the early 1950s, as aircraft engines for passenger flight became more powerful. Unfortunately, there is no way to evade a fundamental scaling law imposed by the laws of mechanics, and it is impossible to make a very-high-speed jet quiet. This was the original impetus to develop high-bypass-ratio turbofan aeroengines, to replace jet engines.

3 Further Acoustic Analogies

The acoustic analogy has been extended in many ways. For example, the Ffowcs Williams–Hawkings equation accounts for boundary effects by means of a vector J_i and a tensor L_{ij} to represent mass and momentum flux through a surface, which may be moving. These terms are given by

$$J_i = \rho(u_i - v_i) + \rho_0 v_i$$

and

$$L_{ij} = \rho u_i(u_j - v_j) + (p - p_0)\delta_{ij} - \tau_{ij},$$

where (v_1, v_2, v_3) is the velocity of the surface and τ_{ij} is the viscous stress tensor. Derivatives of J_i and L_{ij}, with suitable delta functions included to localize them on the surface, are then used as source terms for the acoustic wave equation. The Ffowcs Williams–Hawkings equation has been of enduring importance in the study of aircraft noise and is widely used in computational aeroacoustics.

Variations of Lighthill's equation can be obtained by redistributing terms between the right-hand side, where they are regarded as sources, and the left-hand side, where they are regarded as contributing to the wave propagation operator; different choices of the variable (or the combination of variables) on which the wave operator acts are also possible. Although the equations obtained in this way are always exact, the question of how useful they are, and whether the physical interpretations they embody are "real," has at times been contentious and led to interminable discussion! Among the widely accepted equations are those of Lilley and Goldstein, which place the convective effect of the jet mean flow in the wave operator and modify the source terms on the right-hand side accordingly. Morfey's acoustic analogy is also useful; this provides source terms for sound generation by unsteady dissipation and nonuniformities in density, such as occur in the turbulent mixing of hot and cold fluids.

4 Howe's Vortex Sound Equation

Since the vorticity of a flow field is so important in generating sound, both in a turbulent jet and in other flows, it is desirable to have available an equation that contains it explicitly as a source, rather than implicitly, as in the Lighthill stress tensor. The vorticity $\boldsymbol{\omega}$, defined as the curl of the velocity field, is a measure of the local rotation rate of a flow. In vector notation, $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$, where $\wedge$ indicates the cross product; in components,

$$(\omega_1, \omega_2, \omega_3) = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

An extremely useful equation with vorticity as a source is Howe's vortex sound equation, in which the independent acoustic variable is the total enthalpy defined by

$$B = \int \frac{\mathrm{d}p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2.$$

The symbol B is used to stand for Bernoulli, because this variable appears in Bernoulli's equation. Howe's vortex sound equation is a far-reaching generalization of Bernoulli's equation, and is

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{c^2}\frac{\mathrm{D}}{\mathrm{D}t}\right) - \frac{1}{\rho}\nabla \cdot (\rho \nabla)\right)B = \frac{1}{\rho}\nabla \cdot (\rho \boldsymbol{\omega} \wedge \boldsymbol{u}),$$

where c is the speed of sound and $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u} \cdot \nabla$ is the convective derivative. In the limit of low Mach number, this simplifies to

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)B = \nabla \cdot (\boldsymbol{\omega} \wedge \boldsymbol{u}),$$

and the pressure in the radiated sound field is $p = \rho_0 B$.

Howe's equation is used to calculate the sound produced by the interaction of vorticity with any type of surface. Examples include aircraft fan blades interacting with turbulent inflows, or with the vortices shed by any structure upstream of the blades. An advantage of Howe's equation is that it provides analytical solutions to a number of realistic problems, and hence gives the dependence of the sound field on the design parameters of an aircraft.

5 Current Research in Aircraft Noise

Aircraft noise is such an important factor in limiting the future growth of aviation that research is carried out worldwide to limit its generation. Jet noise is dominant at takeoff, fan and engine noise are dominant in flight, and airframe noise is dominant at landing, so there is plenty for researchers to do. Sources of engine noise are combustion and turbomachinery, and sources of airframe noise are flaps, slats, wing tips, nose gear, and main landing gear. The propagation of sound once it has been generated is also an important area of aircraft noise research, especially the propagation of sonic boom.

A new subject, computational aeroacoustics, was created thirty years ago, and this is now the dominant tool in aircraft noise research. The reader might wonder why its elder sibling, computational fluid dynamics, is not adequate for the task of predicting aircraft noise simply by "adding a little bit of compressibility" to an established code. The answer lies in one of the most pervasive ideas in mathematics, that of an invariant—here, the constant in Bernoulli's theorem. In an incompressible flow, a consequence of Bernoulli's theorem is that pressure fluctuations are entirely balanced by corresponding changes in velocity. Specialized computational techniques must therefore be developed to account for the minute changes in the Bernoulli "constant" produced by fluid compressibility and satisfying the wave equation. Computational aeroacoustics takes full account of such matters, which are responsible for the fact that only a minute proportion of near-field energy (the province of computational fluid dynamics) propagates as sound.

Research on aircraft noise takes place in the world's universities, companies, and government research establishments, and all of it relies heavily on mathematics. It is remarkable how many of the most practical contributions to the subject are made by individuals who are mathematicians; no less than six of the authors in the further reading section below took their first degree in mathematics (many with consummate distinction) and have used mathematics throughout their careers.

Further Reading

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