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Research

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This paper presents a technique, asymptotic decoupling, for analysing wave propagation in multi-layered media. The technique leads to a hierarchy of approximations to the exact dispersion relation, obtained from finite-product approximations to low-order dispersion relations appearing as factors in the asymptotically decoupled limits. Levels of refinement may be added or removed according to the frequency range of interest, the degree of accuracy required, and the material and geometrical parameters of the different layers. This is shown to be particularly useful in stiff problems, because unlimited accuracy is obtainable without redundancy even when Young's moduli and the thicknesses of the layers differ by many orders of magnitude, for example in a stiff sandwich plate with a very soft core. Full details are presented for a non-trivial example, that of antisymmetric waves in a three-layered planar elastic waveguide. Comparisons are made with two widely used approximations, Tiersten's thin-skin approximation and the composite Timoshenko approximation. The mathematical basis of the paper is the asymptotic decoupling of the wave motion in different layers in the limit of indefinitely large or small density ratio.

1. Introduction

This paper is concerned with wave interactions in layered media. Although powerful numerical methods continue to be developed for the analysis of such interactions [1–5], it is found, in practice, that analytical approaches leading to good approximations provide an invaluable adjunct to the numerical results, not least because of their explicit parametric dependence and ease of physical interpretation. However, in deriving these analytical approximations, a fundamental difficulty occurs. This is that the strength of an interaction, which

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may range from being negligible to being of overwhelming importance, depends sensitively on many parameters, most notably the frequency and wavenumber of the wave motion, but also on the material and geometrical parameters of the different layers, for example their density, stiffness, Poisson's ratio and thickness. This sensitivity has led to a proliferation of approximate methods, which have sometimes proved to be of enduring value [6], but which are not always obtained from simple principles and which have ranges of validity that can be difficult or even impossible to interpret. In particular, low-frequency or low-wavenumber approximations offer little clue as to how they would be extended to arbitrarily high frequencies or wavenumbers.

It is shown in this paper that the earlier-mentioned difficulties may be resolved by a hierarchical approach based on two ideas. The first idea is to write the dispersion relation in a form that contains products of dispersion relations for low-dimensional sub-problems. These sub-problems are obtained from limits in which the wave motions in the different layers decouple from each other, because decoupling corresponds to factorization of the dispersion relation. The limits are based on indefinitely large or small density ratios of the component media, because the exact dispersion then becomes progressively better approximated by a product. Thus, the limits give asymptotic decoupling. The second idea is to approximate the sub-problem dispersion relations independently, in order to obtain the simplest approximation that captures the interactions important in a given parameter regime.

The question then arises of what approximations to use for the sub-problem dispersion relations. It is shown that the recently developed finite-product method [7] provides approximations that are ideal. There are two reasons for this. The first reason is that a finite-product approximation allows arbitrarily high accuracy to be attained simply by increasing the number of factors in the products. The second reason is that the resulting finite-product approximations to the complete dispersion relation may be compared systematically with existing widely used approximations, for example Tiersten's thin-skin approximation [8–11] or the composite Timoshenko approximation [12–15]. It is then possible to identify individual terms that one approximation includes but another does not. The physical meaning of these terms is evident from the hierarchical structure of sub-problems being used, and numerical checks then determine which of these terms need to be retained and which may be discarded.

The paper is organized as follows. Section 2 gives the required theory of linear wave propagation in a three-layered elastic waveguide, including full details for antisymmetric waves. In §3, the resulting 6×6 determinant is expressed in a form containing products of dispersion relations for sub-problems, and this form is used to plot the exact dispersion for the full problem. Low- and high-frequency limits are derived in §4. Families of finite-product approximations to the dispersion relation are obtained in §5, where their remarkable numerical accuracy is demonstrated, even for low order; this is the most practically useful part of the paper. Sections 6 and 7 demonstrate the advantages, both analytical and numerical, of these high-accuracy finite products over existing low-order approximations, namely those of Tiersten and of Timoshenko. Conclusions are presented in §8.

2. A three-layered elastic waveguide

(a) Governing equations

An isotropic elastic medium of thickness h occupies a core layer $-\infty < x < \infty$, $|y| < h/2$ lying between two surface layers of thickness h_s composed of a different isotropic elastic medium. The surface layers occupy $h/2 < |y| < h/2 + h_s$, and are in perfectly bonded contact with the core layer at the interfaces $|y| = h/2$. This three-layered structure supports elastic waves that will be assumed to satisfy the linear equations [16, pp. 59, 257; 5]

$$u_{tt} = c_1^2 u_{xx} + (c_1^2 - c_2^2) v_{xy} + c_2^2 u_{yy}, \tag{2.1}$$

$$v_{tt} = c_2^2 v_{xx} + (c_1^2 - c_2^2) u_{xy} + c_1^2 v_{yy}, \tag{2.2}$$

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$$u_{stt} = c_{1s}^2 u_{sxx} + (c_{1s}^2 - c_{2s}^2) v_{sxy} + c_{2s}^2 u_{syy} \tag{2.3}$$

and

$$v_{stt} = c_{2s}^2 v_{sxx} + (c_{1s}^2 - c_{2s}^2) u_{sxy} + c_{1s}^2 v_{syy}. \tag{2.4}$$

In equations (2.1) and (2.2), which refer to the core, $(u(x,y,t), v(x,y,t))$ are longitudinal and transverse displacements at position (x,y) and time t, subscripts x,y and t denote partial differentiation, c_1 is the P-wave speed, i.e. compression-wave speed, and c_2 is the S-wave speed, i.e. shear-wave speed. Equations (2.3) and (2.4) refer to the surface layers; here and throughout the paper, surface-layer variables are indicated by a subscript s. We shall often refer to the surface layers as the skin. The core has Young's modulus E, Poisson's ratio ν, undisturbed density ρ and reference speed $c_0 = (E/\rho)^{1/2}$; the corresponding skin quantities are E_s, ν_s, ρ_s and $c_{0s} = (E_s/\rho_s)^{1/2}$. In plane strain, the wave speeds are

$$c_1 = \left\{ \frac{1-\nu}{(1+\nu)(1-2\nu)} \right\}^{1/2} c_0, \quad c_2 = \frac{c_0}{\{2(1+\nu)\}^{1/2}}, \tag{2.5}$$

and similarly for c_{1s}, c_{2s}. For plane stress, the values of c_1 and c_{1s} are replaced by $c_0/(1-\nu^2)^{1/2}$ and $c_{0s}/(1-\nu_s^2)^{1/2}$. In numerical calculations, we shall take $\nu = \nu_s = 0.3$. Thus, in the core, $c_1 = 1.16c_0$, $c_2 = 0.62c_0$ for plane strain and $c_1 = 1.05c_0$, $c_2 = 0.62c_0$ for plane stress; and similarly in the skin. The normal stresses τ_{xx}, τ_{yy} and shear stress τ_{xy} in the core, corresponding to displacements (u, v), are

$$\tau_{xx} = \rho\{c_1^2 u_x + (c_1^2 - 2c_2^2) v_y\}, \quad \tau_{xy} = \rho c_2^2 (u_y + v_x) \tag{2.6}$$

and

$$\tau_{yy} = \rho\{(c_1^2 - 2c_2^2) u_x + c_1^2 v_y\}, \tag{2.7}$$

and similarly for τ_{sxx}, τ_{sxy} and τ_{syy}.

(b) Antisymmetric waves

We consider antisymmetric waves in the earlier-mentioned three-layered waveguide, i.e. waves in which u and u_s are odd in the transverse coordinate y, whereas v and v_s are even in y. These waves reduce to simple bending waves in the limit of low frequency and wavenumber, but otherwise are of complex structure. We seek solutions of the governing equations with real frequency ω and longitudinal wavenumber k in which all components are proportional to $e^{-i\omega t + ikx}$. In the core, the components are taken to be linear combinations of $\sin ly$ and $\cos ly$; by equations (2.1) and (2.2), the transverse wavenumber l, which may be complex, must satisfy $l^2 = (\omega/c_1)^2 - k^2$ or $l^2 = (\omega/c_2)^2 - k^2$, corresponding to P-waves and S-waves. Similarly, in the skin, the components are taken to be linear combinations of $\sin l_s y_s$ and $\cos l_s y_s$, where l_s, which again may be complex, satisfies $l_s^2 = (\omega/c_{1s})^2 - k^2$ or $l_s^2 = (\omega/c_{2s})^2 - k^2$, by equations (2.3) and (2.4). Here, y_s is a transverse coordinate measured from the centre line of the skin, so that the upper layer of skin has its lower boundary at $y_s = -h_s/2$ and upper boundary at $y_s = h_s/2$. Thus, the line $y = h/2$ is the same as the line $y_s = -h_s/2$, i.e. is the boundary on which the continuity conditions of bonded contact are applied. Because we are considering antisymmetric waves, we shall give formulae only for the core and the upper skin, because the field in the lower skin is determined by the fact that (u_s, v_s) are (odd, even) in y.

(c) Displacement and stress in the core

In the core, the displacement field may be written as [16]

$$\begin{pmatrix} u \\ v \end{pmatrix} = A_1 h \begin{pmatrix} iKL_1^{-1} \sin L_1 Y \\ \cos L_1 Y \end{pmatrix} + A_2 h \begin{pmatrix} -L_2 \sin L_2 Y \\ -iK \cos L_2 Y \end{pmatrix}. \tag{2.8}$$

We use dimensionless variables $Y = y/h$, $K = kh$, and

$$L_1^2 = \left(\frac{\omega h}{c_1}\right)^2 - (kh)^2, \quad L_2^2 = \left(\frac{\omega h}{c_2}\right)^2 - (kh)^2 \quad \text{and} \quad L_3^2 = \left(\frac{\omega h}{\sqrt{2} c_2}\right)^2 - (kh)^2. \tag{2.9}$$

For the moment, A_1 and A_2 are arbitrary. We omit the factor $\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}kx}$ from all field expressions. The stresses (τ_{xy}, τ_{yy}) corresponding to (2.8) are

$$\begin{pmatrix}\tau_{xy}\\ \tau_{yy}\end{pmatrix} = 2\rho c_2^2\left\{A_1\begin{pmatrix}\mathrm{i}K\cos L_1Y\\ -L_3^2L_1^{-1}\sin L_1Y\end{pmatrix} + A_2\begin{pmatrix}-L_3^2\cos L_2Y\\ \mathrm{i}KL_2\sin L_2Y\end{pmatrix}\right\}. \tag{2.10}$$

Throughout this paper, a subscript 1 refers to P-waves, and a subscript 2 refers to S-waves. For complex K, we write $K = K_\mathrm{r} + \mathrm{i}K_\mathrm{i}$; if K is real by implication, then we usually write K rather than K_r.

(d) Displacement and stress in the skin

In the upper skin, the displacement field is

$$\begin{aligned}\begin{pmatrix}u_\mathrm{s}\\ v_\mathrm{s}\end{pmatrix} &= A_{1\mathrm{s}}h_\mathrm{s}\begin{pmatrix}\mathrm{i}K_\mathrm{s}L_{1\mathrm{s}}^{-1}\sin L_{1\mathrm{s}}Y_\mathrm{s}\\ \cos L_{1\mathrm{s}}Y_\mathrm{s}\end{pmatrix} + A_{2\mathrm{s}}h_\mathrm{s}\begin{pmatrix}-L_{2\mathrm{s}}\sin L_{2\mathrm{s}}Y_\mathrm{s}\\ -\mathrm{i}K_\mathrm{s}\cos L_{2\mathrm{s}}Y_\mathrm{s}\end{pmatrix}\\ &\quad + B_{1\mathrm{s}}h_\mathrm{s}\begin{pmatrix}-\mathrm{i}K_\mathrm{s}\cos L_{1\mathrm{s}}Y_\mathrm{s}\\ L_{1\mathrm{s}}\sin L_{1\mathrm{s}}Y_\mathrm{s}\end{pmatrix} + B_{2\mathrm{s}}h_\mathrm{s}\begin{pmatrix}\cos L_{2\mathrm{s}}Y_\mathrm{s}\\ -\mathrm{i}K_\mathrm{s}L_{2\mathrm{s}}^{-1}\cos L_{2\mathrm{s}}Y_\mathrm{s}\end{pmatrix},\end{aligned} \tag{2.11}$$

with arbitrary coefficients $A_{1\mathrm{s}}, A_{2\mathrm{s}}, B_{1\mathrm{s}}, B_{2\mathrm{s}}$ at this stage. Dimensionless variables for the skin are $Y_\mathrm{s} = y_\mathrm{s}/h_\mathrm{s}$, $K_\mathrm{s} = kh_\mathrm{s}$, and

$$L_{1\mathrm{s}}^2 = \left(\frac{\omega h_\mathrm{s}}{c_{1\mathrm{s}}}\right)^2 - (kh_\mathrm{s})^2,\quad L_{2\mathrm{s}}^2 = \left(\frac{\omega h_\mathrm{s}}{c_{2\mathrm{s}}}\right)^2 - (kh_\mathrm{s})^2 \quad\text{and}\quad L_{3\mathrm{s}} = \left(\frac{\omega h_\mathrm{s}}{\sqrt{2}c_{2\mathrm{s}}}\right)^2 - (kh_\mathrm{s})^2. \tag{2.12}$$

The stresses $(\tau_{\mathrm{s}xy}, \tau_{\mathrm{s}yy})$ corresponding to (2.11) are

$$\begin{aligned}\begin{pmatrix}\tau_{\mathrm{s}xy}\\ \tau_{\mathrm{s}yy}\end{pmatrix} &= 2\rho_\mathrm{s}c_{2\mathrm{s}}^2\left\{A_{1\mathrm{s}}\begin{pmatrix}\mathrm{i}K_\mathrm{s}\cos L_{1\mathrm{s}}Y_\mathrm{s}\\ -L_{3\mathrm{s}}^2L_{1\mathrm{s}}^{-1}\sin L_{1\mathrm{s}}Y_\mathrm{s}\end{pmatrix} + A_{2\mathrm{s}}\begin{pmatrix}-L_{3\mathrm{s}}^2\cos L_{2\mathrm{s}}Y_\mathrm{s}\\ \mathrm{i}K_\mathrm{s}L_{2\mathrm{s}}\sin L_{2\mathrm{s}}Y_\mathrm{s}\end{pmatrix}\right.\\ &\quad\left. + B_{1\mathrm{s}}\begin{pmatrix}\mathrm{i}K_\mathrm{s}L_{1\mathrm{s}}\sin L_{1\mathrm{s}}Y_\mathrm{s}\\ L_{3\mathrm{s}}^2\cos L_{1\mathrm{s}}Y_\mathrm{s}\end{pmatrix} + B_{2\mathrm{s}}\begin{pmatrix}-L_{3\mathrm{s}}^2L_{2\mathrm{s}}^{-1}\sin L_{2\mathrm{s}}Y_\mathrm{s}\\ -\mathrm{i}K_\mathrm{s}\cos L_{2\mathrm{s}}Y_\mathrm{s}\end{pmatrix}\right\}.\end{aligned} \tag{2.13}$$

(e) Determinant form of the dispersion relation

The boundary conditions are that $\tau_{\mathrm{s}xy} = 0$ and $\tau_{\mathrm{s}yy} = 0$ at the free outer boundary $Y_\mathrm{s} = \frac{1}{2}$, and that $(u, v, \tau_{xy}, \tau_{yy})$ evaluated at $Y = \frac{1}{2}$ equal $(u_\mathrm{s}, v_\mathrm{s}, \tau_{\mathrm{s}xy}, \tau_{\mathrm{s}yy})$ evaluated at $Y_\mathrm{s} = -\frac{1}{2}$; the latter is the condition of bonded contact at the boundary between the core and the skin. This gives a set of six homogeneous linear equations for the coefficients A_1, A_2, $A_{1\mathrm{s}}$, $A_{2\mathrm{s}}$, $B_{1\mathrm{s}}$, $B_{2\mathrm{s}}$; equating the determinant of the coefficients to zero gives a relation between ω and k, namely the dispersion relation. The complexity of such a 6×6 determinant might be thought to defy analysis; however, the point of this paper is that the dispersion relation has a hierarchical structure, and that much of it can be written down at once from appropriate limits. This idea will now be developed.

3. Asymptotic decoupling

The earlier-defined 6×6 determinant gives a dispersion relation quadratic in the densities (ρ, ρ_s) and of the form

$$(\rho c_2^2 h_\mathrm{s})^2 P + (\rho c_2^2 h_\mathrm{s})(\rho_\mathrm{s} c_{2\mathrm{s}}^2\, h)Q + (\rho_\mathrm{s} c_{2\mathrm{s}}^2\, h)^2 R = 0. \tag{3.1}$$

Here P, Q and R are functions of ω and k, and of the parameters of the problem other than the densities. In what follows, we consider limiting values of ρ/ρ_s with the wave speeds held constant in the limiting process.

(a) Determination of R

The function R is determined by the limit $\rho/\rho_s \to 0$, because in this limit, equation (3.1) reduces to $R = 0$. However, the three-layer problem decouples in the limit: the core effectively acquires a skin of infinite inertia, and thus becomes subject to rigid boundary conditions, whereas the skin effectively becomes adjacent to a massless core, and so both boundaries of the skin layers (not just the outer boundaries) become traction-free. Hence, the limit $R = 0$ must factorize into the product of dispersion relations for the core and the skin separately, i.e. into dispersion relations for separate sub-problems. This implies that R contains factors that may be written in a compact notation as $D_{B'}$, D_{Bs} and D_{Ss}, so that

$$R \propto D_{B'} D_{Bs} D_{Ss}. \tag{3.2}$$

We now determine these factors, explaining why there are three and not two, and at the same time introduce a systematic notation for the various sub-problem dispersion relations. The sub-problems form the irreducible components in our hierarchical scheme.

(i) Decoupled skin

Although only antisymmetric waves are being considered, nevertheless, both symmetric and antisymmetric waves must be included in a skin layer, because antisymmetry arises by transformation between opposite layers. Hence, the dispersion relation for a skin layer decoupled from the core factorizes into the product of that for bending waves in the skin, written as $D_{Bs} = 0$, and that for stretching waves in the skin, written as $D_{Ss} = 0$, with traction-free boundary conditions for both. Henceforth, a subscript B indicates bending, and a subscript S indicates stretching; as previously mentioned, a subscript s indicates the skin.

The trigonometric functions needed in the skin dispersion relations are

$$S_{1s} = \frac{\sin(L_{1s}/2)}{L_{1s}/2}, \quad S_{2s} = \frac{\sin(L_{2s}/2)}{L_{2s}/2}, \quad C_{1s} = \cos\left(\frac{L_{1s}}{2}\right), \quad C_{2s} = \cos\left(\frac{L_{2s}}{2}\right). \tag{3.3}$$

Standard calculations [16] then give

$$D_{Bs} = L_{3s}^4 S_{1s} C_{2s} + K_s^2 L_{2s}^2 C_{1s} S_{2s}, \quad D_{Ss} = K_s^2 L_{1s}^2 S_{1s} C_{2s} + L_{3s}^4 C_{1s} S_{2s}. \tag{3.4}$$

(ii) Decoupled core

The decoupled core contains antisymmetric waves, for which the dispersion relation is $D_{B'} = 0$. A prime is used to indicate rigid boundaries. Thus, by a slight extension of terminology, antisymmetric waves between rigid boundaries are regarded as being a type of constrained bending wave, hence, the use of the subscript B′. The trigonometric functions needed in the core dispersion relation are

$$S_1 = \frac{\sin(L_1/2)}{L_1/2}, \quad S_2 = \frac{\sin(L_2/2)}{L_2/2}, \quad C_1 = \cos\left(\frac{L_1}{2}\right), \quad C_2 = \cos\left(\frac{L_2}{2}\right). \tag{3.5}$$

Standard calculations give

$$D_{B'} = K^2 S_1 C_2 + L_2^2 C_1 S_2. \tag{3.6}$$

(iii) Combined core and skin

Combination of the earlier-mentioned results for the decoupled core and skin problems gives the proportionality relation (3.2) for R. Only the ratios of P, Q and R are determined by (3.1), and it is convenient to fix values by the choice of Q given below. With this choice, the constant of proportionality in relation (3.2) is −2. Hence, the result of the asymptotic decoupling analysis for the limit $\rho/\rho_s \to 0$ is the exact expression

$$R = -2 D_{B'} D_{Bs} D_{Ss}. \tag{3.7}$$

(b) Determination of P

In a similar way, the function P in equation (3.1) is determined by the limit $\rho_{\mathrm{s}}/\rho \to 0$, because in this limit the equation reduces to $P = 0$. The decoupling in the three-layer problem is now such that core has a massless skin, so that its boundaries become traction-free; and each skin is bonded to a core of infinite inertia, so that its inner boundary becomes subject to rigid boundary conditions. The outer boundary of each skin remains traction-free. Hence, P contains factors that will be written as D_{B} and D_{BSs}, and

$$P \propto D_{\mathrm{B}} D_{\mathrm{BSs}}. \tag{3.8}$$

These factors will now be determined, and the notation D_{BSs} explained.

(i) Decoupled core

The decoupled core contains ordinary antisymmetric waves, i.e. of bending-wave type with traction-free boundary conditions. Thus, the dispersion relation is $D_{\mathrm{B}} = 0$, where

$$D_{\mathrm{B}} = L_3^4 S_1 C_2 + K^2 L_2^2 C_1 S_2. \tag{3.9}$$

(ii) Decoupled skin

The skin problem is of mixed type, because of the different boundary conditions on the inner and outer boundaries. Thus, although the skin is decoupled from the core, within the skin there is strong coupling of antisymmetric and symmetric waves, i.e. waves of bending and stretching type. Hence, the dispersion relation is $D_{\mathrm{BSs}} = 0$, where [17]

$$D_{\mathrm{BSs}} = D_{\mathrm{Bs}} D_{\mathrm{S's}} + D_{\mathrm{Ss}} D_{\mathrm{B's}} - \tfrac{1}{2} \Omega_{2\mathrm{s}}^4. \tag{3.10}$$

Here, $D_{\mathrm{S's}}$ is defined so that $D_{\mathrm{S's}} = 0$ is the dispersion relation for symmetric waves in an elastic layer between rigid boundaries; these waves are a type of constrained stretching wave. Similarly, $D_{\mathrm{B's}}$ is defined by analogy with $D_{\mathrm{B'}}$ and describes a constrained bending wave. Hence

$$D_{\mathrm{S's}} = L_{1\mathrm{s}}^2 S_{1\mathrm{s}} C_{2\mathrm{s}} + K_{\mathrm{s}}^2 C_{1\mathrm{s}} S_{2\mathrm{s}}, \quad D_{\mathrm{B's}} = K_{\mathrm{s}}^2 S_{1\mathrm{s}} C_{2\mathrm{s}} + L_{2\mathrm{s}}^2 C_{1\mathrm{s}} S_{2\mathrm{s}}. \tag{3.11}$$

The quantity $\Omega_{2\mathrm{s}}$ in equation (3.10) is a dimensionless frequency. The definitions of this and related dimensionless frequencies needed later are

$$\Omega_1 = \frac{\omega h}{c_1}, \quad \Omega_2 = \frac{\omega h}{c_2}, \quad \Omega_{1\mathrm{s}} = \frac{\omega h_{\mathrm{s}}}{c_{1\mathrm{s}}}, \quad \Omega_{2\mathrm{s}} = \frac{\omega h_{\mathrm{s}}}{c_{2\mathrm{s}}}. \tag{3.12}$$

(iii) Combined core and skin

Combination of the above results gives the proportionality relation (3.8); with the choice of Q below, the constant of proportionality is -1. Hence, the result of the asymptotic decoupling analysis for the limit $\rho_{\mathrm{s}}/\rho \to 0$ is the exact expression

$$P = -D_{\mathrm{B}} D_{\mathrm{BSs}}. \tag{3.13}$$

(c) Determination of Q

Determination of the interaction function Q in equation (3.1) requires the full 6×6 determinant specified in §2*e*. A hand calculation, based on 2×2 minors and their cofactors, or a computer calculation using a symbolic manipulation program such as MATHEMATICA, leads to quantities of dispersion-relation type defined by

$$D_{\mathrm{M}} = L_3^2 S_1 C_2 - L_2^2 C_1 S_2, \quad D_{\mathrm{N}} = L_1^2 S_1 C_2 - L_3^2 C_1 S_2 \tag{3.14}$$

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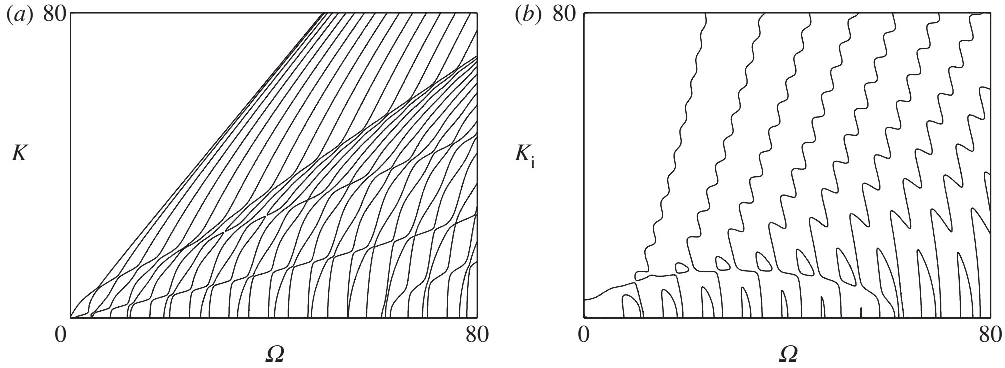

Figure 1. Dispersion relation for antisymmetric waves in a three-layered elastic waveguide with $h_s/h = \frac{1}{10}$, $\rho_s/\rho = 10$, $E_s/E = 100$ and $\nu = \nu_s = 0.3$. (*a*) Real branches. (*b*) Imaginary branches. The horizontal axis is $\Omega = \omega h/c_0$, where $c_0 = (E/\rho)^{1/2}$.

and

$$D_{Ms} = L_{3s}^2 S_{1s} C_{2s} - L_{2s}^2 C_{1s} S_{2s}, \quad D_{Ns} = L_{1s}^2 S_{1s} C_{2s} - L_{3s}^2 C_{1s} S_{2s}, \tag{3.15}$$

and to full interaction terms defined by

$$I_C = C_1 C_2 C_{1s} C_{2s} + \frac{1}{16} L_2^2 L_{1s}^2 S_1 S_2 S_{1s} S_{2s} \tag{3.16}$$

and

$$I_S = \frac{1}{4}(L_2^2 S_1 S_2 C_{1s} C_{2s} + L_{2s}^2 C_1 C_2 S_{1s} S_{2s}). \tag{3.17}$$

In terms of these quantities, the interaction function Q is

$$Q = \Omega_2^2 \Omega_{2s}^2 (D_{Bs} I_C - D_{Ss} I_S) - 2KK_s D_M (D_{Bs} D_{Ns} - D_{Ss} D_{Ms}). \tag{3.18}$$

(d) The full dispersion relation

Equation (3.1), with P, Q and R given by the factorized expressions in equations (3.13), (3.18) and (3.7), is the hierarchical form of the dispersion relation for antisymmetric waves in the three-layered elastic waveguide. The left-hand side is to be regarded as primarily a function of frequency ω and longitudinal wavenumber k; the many other quantities appearing in its definition are geometrical and material parameters. A non-trivial check of the expressions for P, Q and R is evaluation and simplification of the left-hand side of equation (3.1) in the special case of identical media in the layers, but arbitrary thicknesses h and h_s. The result must be an expression of the functional form of D_B or D_{Bs}, but with all quantities evaluated for thickness $h + 2h_s$. Extended use of trigonometric identities verifies this result. Plots of the real and imaginary branches of equation (3.1) are given in figure 1*a*,*b* for plane strain with Poisson's ratio $\nu = \nu_s = 0.3$ and $h_s/h = \frac{1}{10}$, $\rho_s/\rho = 10$, $E_s/E = 100$. The horizontal axis is the dimensionless frequency $\Omega = \omega h/c_0$, where $c_0 = (E/\rho)^{1/2}$. Large-scale features of the plots, for example the arc evident in figure 1*b*, scale with the skin parameters.

4. Limiting forms of the dispersion relation

(a) Low frequency

An important aspect of the full dispersion relation (3.1) is its limiting form at low frequency and wavenumber, for arbitrary values of all geometrical and material parameters. This Euler–Bernoulli limit is determined by Taylor-series expansion of equation (3.1), with the dimensionless frequencies and wavenumbers taken to be much less than unity. A minor complication is that equation (3.1) is exactly divisible by ω^6, in the sense that the left-hand side is the product of ω^6 and an analytic function of ω (although the analytic function cannot be written down in a simple way); this occurs because $D_{\mathrm{B}}, \ldots, D_{\mathrm{Ns}}$ are all divisible by ω^2 in the same sense, and D_{BSs} is likewise divisible by ω^4. Hence, the expansions of P, Q and R contain a factor of ω^6 that must be cancelled out before the Euler–Bernoulli limit is revealed.

It is convenient to define bending wave characteristic speeds (c_{B}, c_{Bs}) and corresponding dimensionless frequencies (Ω_{B}, Ω_{Bs}) by

$$c_{\mathrm{B}}^2 = \frac{c_2^2(c_1^2 - c_2^2)}{3c_1^2}, \quad c_{\mathrm{Bs}}^2 = \frac{c_{2\mathrm{s}}^2(c_{1\mathrm{s}}^2 - c_{2\mathrm{s}}^2)}{3c_{1\mathrm{s}}^2}, \quad \Omega_{\mathrm{B}} = \frac{\omega h}{c_{\mathrm{B}}}, \quad \Omega_{\mathrm{Bs}} = \frac{\omega h_{\mathrm{s}}}{c_{\mathrm{Bs}}}. \tag{4.1}$$

These definitions are such that the dispersion relations for Euler–Bernoulli bending waves in the core and skin separately would be $\Omega_{\mathrm{B}}^2 = K^4$ and $\Omega_{\mathrm{Bs}}^2 = K_{\mathrm{s}}^4$. The first terms in the Taylor-series expansion of P, Q and R are

$$P = \frac{1}{72}(\Omega_2^2 - \Omega_1^2)(\Omega_{\mathrm{B}}^2 - K^4)(\Omega_{2\mathrm{s}}^2 - \Omega_{1\mathrm{s}}^2)\Omega_{\mathrm{Bs}}^2 + \cdots, \tag{4.2}$$

$$Q = \frac{1}{12}\Omega_2^2\Omega_{2\mathrm{s}}^2(\Omega_{2\mathrm{s}}^2 - \Omega_{1\mathrm{s}}^2)(\Omega_{\mathrm{Bs}}^2 - 4K_{\mathrm{s}}^4 - 6K_{\mathrm{s}}^3K - 3K_{\mathrm{s}}^2K^2) + \cdots \tag{4.3}$$

and

$$R = -\frac{1}{72}\Omega_2^2(\Omega_{\mathrm{Bs}}^2 - 12K_{\mathrm{s}}^2)(\Omega_{2\mathrm{s}}^2 - \Omega_{1\mathrm{s}}^2)^2(\Omega_{\mathrm{Bs}}^2 - K_{\mathrm{s}}^4) + \cdots. \tag{4.4}$$

Here, the common factors of ω^6 are implicit in the total powers of the dimensionless frequencies, and are present also in all subsequent terms indicated by the dots. Substitution of these expressions for P, Q and R into (3.1), followed by division by ω^6 and simplification, gives at leading order

$$(\rho c_{\mathrm{B}}^2 h_{\mathrm{s}})(\Omega_{\mathrm{B}}^2 - K^4) + 2(\rho_{\mathrm{s}} c_{\mathrm{Bs}}^2 h)(\Omega_{\mathrm{Bs}}^2 - 4K_{\mathrm{s}}^4 - 6K_{\mathrm{s}}^3K - 3K_{\mathrm{s}}^2K^2) = 0. \tag{4.5}$$

This is the Euler–Bernoulli limit of equation (3.1), involving terms only in ω^2 and k^4. The quantity R does not enter into the Euler–Bernoulli limit, as its leading power of dimensionless frequency is higher than that in P and Q. Re-grouping the terms in (4.5) and returning to dimensional variables give

$$(\rho h + 2\rho_{\mathrm{s}} h_{\mathrm{s}})\omega^2 = \{\rho c_{\mathrm{B}}^2 h^3 + \rho_{\mathrm{s}} c_{\mathrm{Bs}}^2[(h + 2h_{\mathrm{s}})^3 - h^3]\}k^4. \tag{4.6}$$

This form expresses the balance between inertia and bending stiffness in the three layers taken as a whole, accounting for the bonded contact.

The Euler–Bernoulli limit represented by equations (4.5) and (4.6) is valid in only a very small region of the (frequency, wavenumber) plane, namely the lower-left corners of the plots in figure 1 up to about $\Omega = 0.01$ and $K = 0.1$ or $K_{\mathrm{i}} = 0.1$. This is clear also in figure 2*a*, where the start of the straight-line segment indicates where the bending regime ends and the stretching regime begins. These results extend those in Liu & Bhattacharya [4] and Lutianov & Rogerson [18].

(b) High frequency

At high frequencies and wavenumbers, the waveguide supports surface Rayleigh waves decoupled from interfacial Stoneley waves. This corresponds to the limit of increasingly large imaginary values of the transverse wavenumbers ($L_1, L_2, L_{1\mathrm{s}}, L_{2\mathrm{s}}$). In the exact dispersion relation, the trigonometric terms then give rise to a common dominant exponential factor, which may be cancelled. Hence, the limiting form of the dispersion relation is obtained from the exact dispersion relation on replacing ($C_1, C_2, C_{1\mathrm{s}}, C_{2\mathrm{s}}$) by $\frac{1}{2}$, and ($S_1, S_2, S_{1\mathrm{s}}, S_{2\mathrm{s}}$) by ($\mathrm{i}/L_1, \mathrm{i}/L_2, \mathrm{i}/L_{1\mathrm{s}}, \mathrm{i}/L_{2\mathrm{s}}$); the

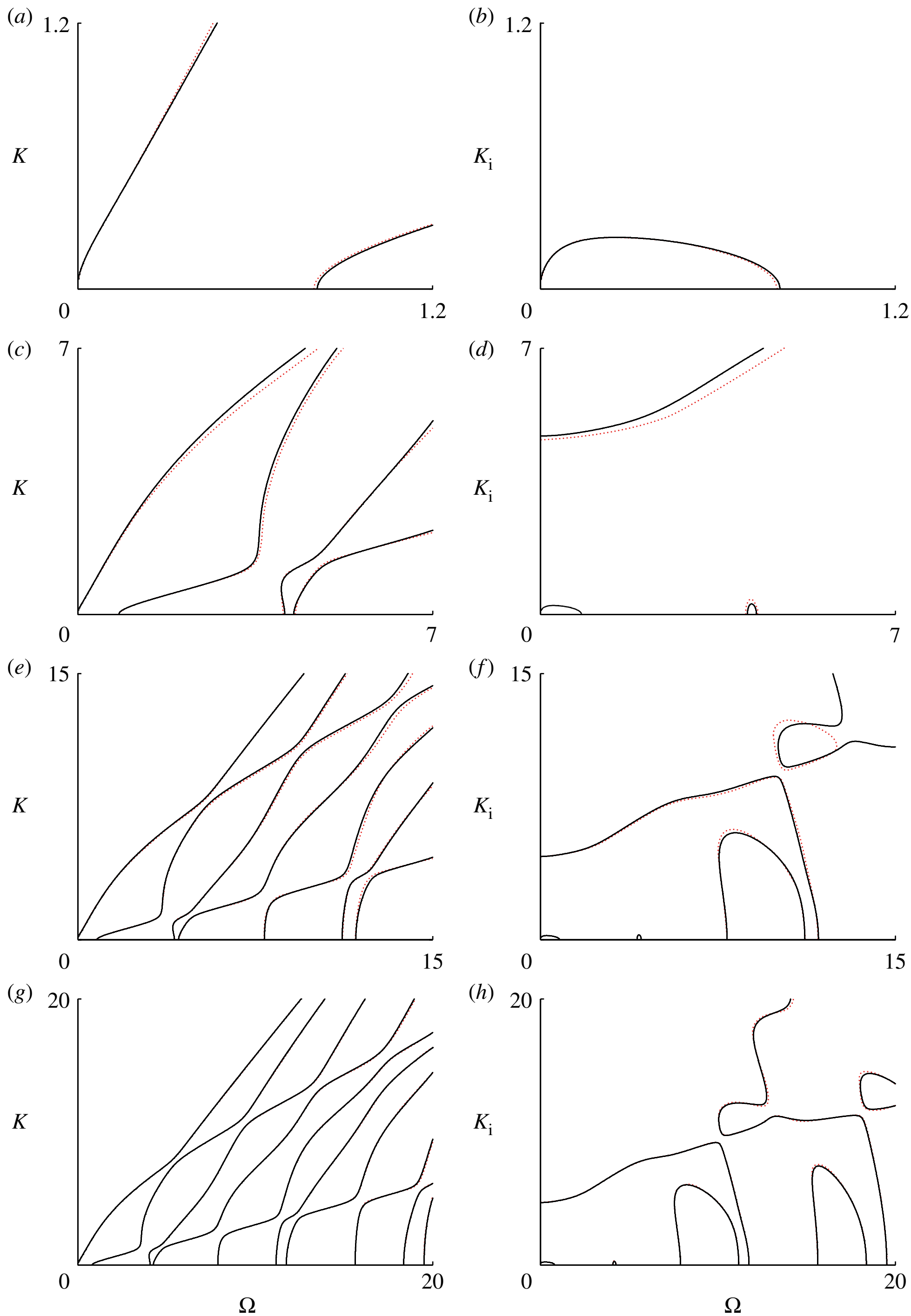

Figure 2. Real and imaginary branches of the dispersion relation for $h_s/h = \frac{1}{10}$, $\rho_s/\rho = 10$, $E_s/E = 100$ and $\nu = \nu_s = 0.3$. Solid lines denote exact dispersion curves; dotted lines denote (m_s, n_s, m, n) finite-product approximations for the values (*a*,*b*) (0, 0, 0, 1); (*c*,*d*) (0, 1, 1, 2); (*e*,*f*) (1, 2, 3, 4); and (*g*,*h*) (3, 4, 7, 8). (Online version in colour.)

term Ω_{2s}^4 in D_{BSs}, lacking an exponential factor, makes no contribution. The result takes the form of equation (3.1), but with the coefficients (P, Q, R) replaced by $(\bar{P}, \bar{Q}, \bar{R})$, where

$$\bar{P} = D_{\mathrm{R}} D_{\mathrm{R's}}, \quad \bar{Q} = \tfrac{1}{4}\Omega_2^2 \Omega_{2s}^2 I_{\mathrm{L}} - 2KK_s D_{\mathrm{R''}} D_{\mathrm{R''s}}, \quad \bar{R} = D_{\mathrm{R'}} D_{\mathrm{Rs}}. \tag{4.7}$$

The core and surface-layer terms are

$$D_{\mathrm{R}} = L_3^4 + K^2 L_1 L_2, \quad D_{\mathrm{R}'} = L_1 L_2 + K^2, \quad D_{\mathrm{R}''} = L_2^2 - L_1 L_2 \tag{4.8}$$

and

$$D_{\mathrm{Rs}} = L_{3\mathrm{s}}^4 + K_{\mathrm{s}}^2 L_{1\mathrm{s}} L_{2\mathrm{s}}, \quad D_{\mathrm{R's}} = L_{1\mathrm{s}} L_{2\mathrm{s}} + K_{\mathrm{s}}^2, \quad D_{\mathrm{R''s}} = L_{2\mathrm{s}}^2 - L_{1\mathrm{s}} L_{2\mathrm{s}}, \tag{4.9}$$

and the interaction term is $I_{\mathrm{L}} = L_1 L_{2\mathrm{s}} + L_2 L_{1\mathrm{s}}$. Here, $D_{\mathrm{R}} = 0$ and $D_{\mathrm{Rs}} = 0$ are the Rayleigh-wave dispersion relations for the core medium and surface-layer medium; similarly, $D_{\mathrm{R}'} = 0$ and $D_{\mathrm{R's}} = 0$ are dispersion-type equations with no real roots, corresponding to the fact that a surface wave cannot occur at a rigid boundary.

In arriving at $(\bar{P}, \bar{Q}, \bar{R})$ from (P, Q, R), a common factor D_{Rs} that appears in all three terms has not been written in equations (4.7); this factor corresponds to the surface Rayleigh wave. The remaining dispersion relation, i.e. equation (3.1) but with coefficients $(\bar{P}, \bar{Q}, \bar{R})$, is the Stoneley-wave dispersion relation. Two simple checks of its correctness are its symmetry with respect to the core and surface media, and the fact that h and h_{s} may be cancelled out: surface and interfacial dispersion relations cannot depend on layer depths. The factors in $\bar{P}$ and $\bar{R}$ correspond to large and small limiting values of the density ratio ρ/ρ_{s}, i.e. asymptotic decoupling. In $\bar{P}$ and $\bar{R}$, the dense-medium factor gives a Rayleigh wave, and the light-medium factor gives what might be called the non-Rayleigh-wave equation, i.e. $D_{\mathrm{R}'} = 0$ or $D_{\mathrm{R's}} = 0$. The Stoneley-wave dispersion relation as given here agrees with that in Ewing *et al*. [19, eqns 3.137–3.139] and Brekhovskikh [20, eqn 8.15]; these references order the terms less systematically, and neither the decoupling in $\bar{P}$ and $\bar{R}$ nor the factors in $\bar{Q}$ are explicit.

5. Finite-product approximations

(a) Alternatives to elasticity theory

The theory of elasticity, as used earlier, is unwieldy for many purposes. An alternative, associated with the work of Timoshenko, Mindlin and Tiersten, is to assume a low-order polynomial dependence of the displacement field on the transverse coordinate. This approach, an extension of Kirchhoff–Love theory, has two limitations. The first is that it does not determine the range of validity of an approximation when applied to higher frequencies and wavenumbers; for example, it does not explain why approximations for antisymmetric waves are accurate up to higher frequencies than one would expect, whereas the opposite is true for symmetric waves. The second limitation is that the approach does not yield a sequence of progressively more accurate approximations, except in a small region of the (frequency, wavenumber) plane; this is a consequence of the finite radius of convergence of the Taylor-series expansions of the field variables in powers of the frequency and wavenumber.

The earlier-mentioned limitations are circumvented by finite-product approximations. Their mathematical theory, based on gamma functions and the cancelling-out of Runge's phenomenon, determines precisely their range of validity in the (frequency, wavenumber) plane [7]; moreover, finite products provide a rapidly convergent sequence of approximations in any finite region in this plane, no matter how great the size of the region, yet introduce no spurious branches into the dispersion relation in any approximation. The lowest-order members of a family of finite-product approximations recover the traditional dispersion relations referred to above [21]. Accordingly, we now present the finite-product theory of the dispersion relation (3.1) for antisymmetric waves in a three-layered elastic waveguide, within the asymptotic decoupling scheme. The approximations derived are hierarchical in two ways: the number of factors in a product may be increased or decreased, to move up or down the hierarchy; and expressions for individual layers may be approximated independently in the sub-problem dispersion relations. For example, if it is known on physical grounds that a stretching wave in the skin is important in a given frequency regime, then one would ensure that this motion is well represented in the corresponding sub-problem approximation.

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(b) Finite products

The exact expressions for $S_1, C_1, S_{1s}, C_{1s}, \ldots$ defined in §3 may be written in terms of the functions S and C defined by $S(s^2) = s^{-1}\sin(s)$ and $C(c^2) = \cos(c)$, or equivalently $S(s) = s^{-1/2}\sin(s^{1/2})$ and $C(c) = \cos(c^{1/2})$. Thus $S_1 = S(L_1^2/4)$, $C_1 = C(L_1^2/4)$, $S_{1s} = S(L_{1s}^2/4)$, $C_{1s} = C(L_{1s}^2/4), \ldots$. Finite products are obtained by truncation of the exact infinite-product representations [22]

$$S(s) = \prod_{m'=1}^{\infty}\left(1 - \frac{s}{(m'\pi)^2}\right), \quad C(c) = \prod_{n'=1}^{\infty}\left(1 - \frac{c}{\{(n'-1/2)\pi\}^2}\right). \tag{5.1}$$

The argument of a finite product is one of $L_1^2/4$, $L_2^2/4$, $L_{1s}^2/4$ or $L_{2s}^2/4$, and will be indicated by a subscript. Thus, we write

$$S_{1m} = \prod_{m'=1}^{m}\left(1 - \frac{L_1^2/4}{(m'\pi)^2}\right), \quad C_{1n} = \prod_{n'=1}^{n}\left(1 - \frac{L_1^2/4}{\{(n'-1/2)\pi\}^2}\right), \tag{5.2}$$

and similarly for (S_{2m}, C_{2n}), (S_{1sm_s}, C_{1sn_s}) and (S_{2sm_s}, C_{2sn_s}) with arguments $L_2^2/4$, $L_{1s}^2/4$ and $L_{2s}^2/4$. Here, (m, n) and (m_s, n_s) refer to the numbers of factors in the (sine, cosine)-based finite products for the core and skin. An empty product, e.g. for m_s or n_s equal to zero, is defined to have the value 1. We shall write the numbers of factors in the order (m_s, n_s, m, n). The idea of the finite-product method is to choose the numbers of factors to be as low as possible consistent with obtaining the required accuracy in the resulting polynomial approximation to the dispersion relation. High accuracy for low (m_s, n_s, m, n) is easily attained, because Runge's phenomenon so readily cancels out.

The quantities introduced in §3 all have finite-product forms, defined for example by

$$D_{Bmn} = L_3^4 S_{1m} C_{2n} + K^2 L_2^2 C_{1n} S_{2m}, \tag{5.3}$$

$$D_{Bsm_sn_s} = L_{3s}^4 S_{1sm_s} C_{2sn_s} + K_s^2 L_{2s}^2 C_{1sn_s} S_{2sm_s} \tag{5.4}$$

and

$$I_{Sm_sn_smn} = \frac{1}{4}(L_2^2 S_{1m} S_{2m} C_{1sn_s} C_{2sn_s} + L_{2s}^2 C_{1n} C_{2n} S_{1sm_s} S_{2sm_s}). \tag{5.5}$$

These lead to finite products $P_{m_sn_smn}, Q_{m_sn_smn}$ and $R_{m_sn_smn}$, and hence to the (m_s, n_s, m, n) finite-product approximation to the dispersion relation (3.1).

(c) The main sequence

Our aim is to find small values of (m_s, n_s, m, n) that capture branches of the dispersion relation up to high frequencies and wavenumbers. It follows from Chapman & Sorokin [7] that this is best achieved by taking $n_s = m_s + 1$ and $n = m + 1$, although the choices $n_s = m_s$ and $n = m$ are also good. This gives a two-parameter family of approximations $(m_s, m_s + 1, m, m + 1)$. It is an advantage to choose a greater number of factors for softer, thicker media, because such media support higher wavenumbers at a given frequency. Good results are obtained by taking n to be a fixed multiple of n_s. Thus, we may take $n = an_s$, where a depends on the softness and thickness of the core relative to the skin. The result is a one-parameter family $(m_s, m_s + 1, am_s + a - 1, am_s + a)$ labelled by the parameter m_s, which we call the main sequence of approximations. For example, $a = 2$ gives $(m_s, m_s + 1, 2m_s + 1, 2m_s + 2)$ for $m_s = 0, 1, \ldots$; this is the sequence $(0, 1, 1, 2)$, $(1, 2, 3, 4)$, $(2, 3, 5, 6), \ldots$. However, the lower values $(0, 0, 0, 1)$, corresponding to the $(0, 0)$ approximation in the skin and the $(0, 1)$ approximation in the core, are also very good, and as noted below, even $(0, 0, 0, 0)$ is good for many purposes.

(d) Numerical results

To provide a challenging test of the numerical accuracy of the earlier-mentioned finite-product approximations within the asymptotic decoupling scheme, we consider a stiff sandwich plate for which the skin, in comparison with the core, has Young's modulus 100 times as great, density 10 times as great and thickness one-tenth as great. In the notation of §2*a*, this is $E_s/E = 100$, $\rho_s/\rho = 10$ and $h_s/h = \frac{1}{10}$; we also take equal Poisson's ratios $\nu_s = \nu = 0.3$.

The real and imaginary branches of the finite-product approximations $(m_s, n_s, m, n) = (0,0,0,1), (0,1,1,2), (1,2,3,4)$ and $(3,4,7,8)$ for real frequency are plotted as dotted lines in figure 2*a*–*h*. The exact dispersion relation is superposed as solid lines, for detailed comparison of finite-product approximations with the exact results of §3. The frequency variable is $\Omega = \omega h/c_0$, where $c_0 = (E/\rho)^{1/2}$ is a reference speed, and the wavenumber variable is $K = kh$. On the plots of imaginary branches, the wavenumber axis is K_i, where $K = iK_i$. Thus, we use 'core units', in which frequencies of order one correspond to natural vibrations of the thick soft core. Large-scale features, which emerge over dimensionless frequency ranges of about 60, correspond to natural vibrations of the thin stiff skin. Such vibrations are implicit in the dispersion relations for the sub-problems identified in §3.

The $(0,0,0,1)$ approximation, shown in figure 2*a*,*b*, has error of at most 1 per cent in almost all of a square region defined by Ω up to 1.2 and K or K_i up to 1.2. The same accuracy is achieved by the $(0,1,1,2), (1,2,3,4)$ and $(3,4,7,8)$ approximations in square regions of size 7, 15 and 20, as shown in figure 2*c*–*h*. The rapid increase in the size of these regions, for modest increases in the number of finite-product factors, indicates the high accuracy provided by finite-product approximations; and there are no spurious branches.

Complex branches in three-dimensional (Ω, K_r, K_i) space, which we do not plot here, are also given accurately by the earlier-mentioned finite-product approximations. The $(0,0,0,0)$ approximation, also not plotted, is similar to the $(0,0,0,1)$ approximation shown in figure 2*a*,*b*, differing only in that it gives frequencies too high by about 8 per cent on the branch that cuts on close to $\Omega = 0.8$. This inaccuracy is easily eliminated by means of a Timoshenko-type correction factor $\kappa = \pi^2/12$. Thus, the $(0,0,0,0)$ approximation contains no fundamental error in its structure, and is noteworthy as the simplest finite-product approximation which is basically correct.

6. Asymptotic decoupling and Tiersten's thin surface approximation

In this section and in the following, we compare the present work with previous approximations that are available when the surface elastic layer is thin [8–11,23,24]. These determine the combined wave motion of a surface layer and an elastic substrate to which it is bonded. The most widely used such method is that of Tiersten [8]. We now show that the main result of Tiersten's method may be cast into an asymptotically decoupled form, and that this makes possible an increase in its range of validity by revealing factors that may be replaced by finite products.

(a) Tiersten's method applied to a three-layered waveguide

In Tiersten's method applied to the problem specified in §2, the surface-layer displacement $(u_s(x, y_s, t), v_s(x, y_s, t))$ is written as

$$u_s = u_{s0} + y_s u_{s1}, \quad v_s = v_{s0} + y_s v_{s1} + y_s^2 v_{s2}, \tag{6.1}$$

where $(u_{s0}, u_{s1}, v_{s0}, v_{s1}, v_{s2})$ are functions of x and t. The stress–strain relation gives

$$u_{s1} = -v_{s0x}, \quad v_{s1} = -\frac{c_{1s}^2 - 2c_{2s}^2}{c_{1s}^2} u_{s0x} \quad \text{and} \quad v_{s2} = -\frac{c_{1s}^2 - 2c_{2s}^2}{2c_{1s}^2} v_{s0xx}, \tag{6.2}$$

so that the displacement is expressible in terms of u_{s0} and v_{s0}. This leads to stretching and bending equations for the surface layer in the form

$$\rho_s h_s u_{s0tt} = 12\rho_s c_{Bs}^2 h_s u_{s0xx} - \tilde{\tau}_{y_s x} \tag{6.3}$$

and

$$\rho_s h_s v_{s0tt} = -\rho_s c_{Bs}^2 h_s^3 v_{s0xxxx} - \tilde{\tau}_{y_s y_s} + \tfrac{1}{2} h_s \tilde{\tau}_{y_s x, x}. \tag{6.4}$$

Here, $\tilde{\tau}_{y_s y_s}$ and $\tilde{\tau}_{y_s x}$ are the normal stress and shear stress on the interface $y_s = -h_s/2$, and $\tilde{\tau}_{y_s x,x}$ is the longitudinal derivative of $\tilde{\tau}_{y_s x}$. Final subscripts on displacement quantities denote partial derivatives.

Equations (6.3) and (6.4) have solutions with u_{s0}, v_{s0}, $\tilde{\tau}_{y_s x}$, $\tilde{\tau}_{y_s y_s}$ proportional to $e^{-i\omega t + ikx}$. Omitting this factor from field expressions, we may take

$$u_{s0} = A_{1s}, \quad v_{s0} = A_{2s}, \tag{6.5}$$

where A_{1s} and A_{2s} are constants, and

$$\tilde{\tau}_{y_s x} = \rho_s h_s(\omega^2 - 12c_{Bs}^2 k^2) A_{1s} \tag{6.6}$$

and

$$\tilde{\tau}_{y_s y_s} = \tfrac{1}{2} i \rho_s h_s^2 k(\omega^2 - 12c_{Bs}^2 k^2) A_{1s} + \rho_s h_s(\omega^2 - c_{Bs}^2 h_s^2 k^4) A_{2s}. \tag{6.7}$$

The skin quantities $(u_s, v_s, \tilde{\tau}_{y_s x}, \tilde{\tau}_{y_s y_s})$, with (u_s, v_s) evaluated at $y_s = -h_s/2$, are continuous with the core quantities $(u, v, \tau_{xy}, \tau_{yy})$ evaluated at $y = h/2$, as given in terms of the arbitrary constants A_1 and A_2 by equations (2.8) and (2.10). In Tiersten's method, when equations (6.1) are evaluated at $y_s = -h_s/2$ for matching purposes, the terms v_{s1} and v_{s2} may be ignored, so that for matching we take

$$u_s = A_{1s} + \tfrac{1}{2} i h_s k A_{2s}, \quad v_s = A_{2s}. \tag{6.8}$$

Thus, continuity of the displacements and tractions leads to a set of four homogeneous linear equations in the coefficients A_1, A_2, A_{1s}, A_{2s}. Equating the determinant of the coefficients to zero gives the Tiersten dispersion relation between ω and k.

(b) Asymptotic decoupling

The dispersion relation just defined is of the form

$$(\rho c_2^2 h_s)^2 \tilde{P} + (\rho c_2^2 h_s)(\rho_s c_{Bs}^2 h) \tilde{Q} + (\rho_s c_{Bs}^2 h)^2 \tilde{R} = 0, \tag{6.9}$$

where $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ are Tiersten analogues of the functions P, Q and R introduced in equation (3.1), and c_{Bs}^2 replaces the earlier c_{2s}^2. Evaluation of the determinant leading to equation (6.9) gives

$$\tilde{P} = 2D_B, \quad \tilde{R} = -\tfrac{1}{2} D_{B'} \tilde{D}_{Bs} \tilde{D}_{Ss} \tag{6.10}$$

and

$$\tilde{Q} = \Omega_2^2 C_1 C_2 \tilde{D}'_{Bs} - \tfrac{1}{4} \Omega_2^2 L_2^2 S_1 S_2 \tilde{D}_{Ss} - K K_s D_M \tilde{D}_{Bs}, \tag{6.11}$$

where

$$\tilde{D}_{Bs} = \Omega_{Bs}^2 - K_s^4, \quad \tilde{D}_{Ss} = \Omega_{Bs}^2 - 12K_s^2, \quad \tilde{D}'_{Bs} = \Omega_{Bs}^2 - 4K_s^4 + \tfrac{1}{4} \Omega_{2s}^2 K_s^2. \tag{6.12}$$

Thus, $\tilde{D}_{Bs}$ and $\tilde{D}_{Ss}$ are the Euler–Bernoulli and Poisson analogues of D_{Bs} and D_{Ss} for bending waves and stretching waves; and the first two terms of $\tilde{D}'_{Bs}$ correspond to terms in the Taylor-series expression (4.3) for Q. Hence, equations (6.10)–(6.12) make explicit the way in which the exact dispersion is modified term by term in Tiersten's approximation. For example, the terms C_1C_2 and S_1S_2 in equation (6.11) for $\tilde{Q}$ derive from I_C and I_S in equations (3.16) and (3.17), as used in equation (3.18) for Q. Similarly, comparison of equation (3.13) for P with equation $(6.10)_1$ for $\tilde{P}$ shows that D_{BSs} as defined in equation (3.10) is approximated by a term proportional to ω^4. In making these comparisons, a factor proportional to ω^4 needs to be cancelled from the approximations to the exact expressions P, Q, R to arrive at $\tilde{P}, \tilde{Q}, \tilde{R}$, in accord with the remarks in §4*a*.

A check of equations (6.10)–(6.12) is provided by their limiting form at low frequency and wavenumber. The resulting series agrees with the displayed terms in equations (4.2)–(4.4) obtained from the exact dispersion relation. Tracking individual powers of frequency and wavenumber through the calculation gives another way of linking (P, Q, R) with $(\tilde{P}, \tilde{Q}, \tilde{R})$ term by term.

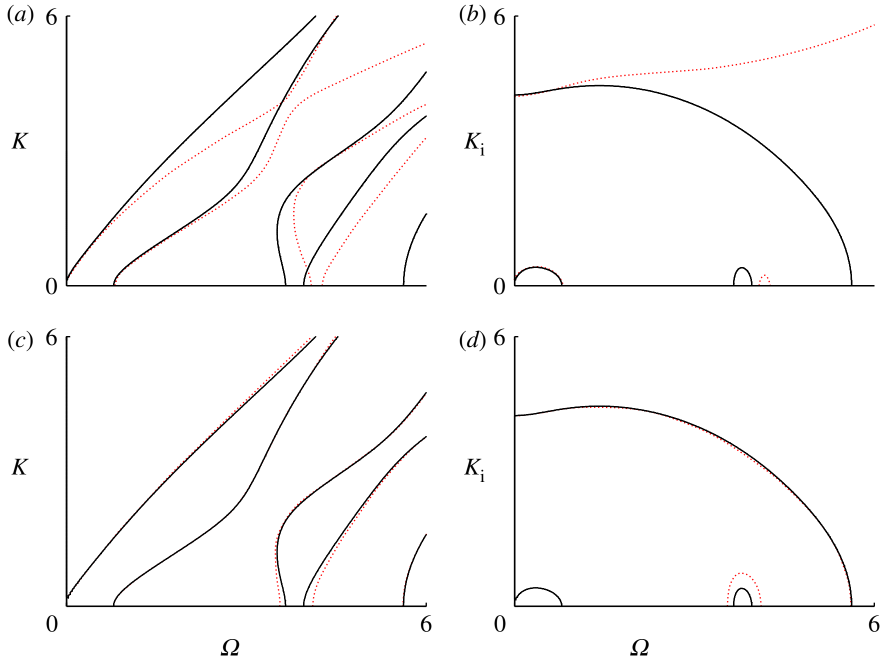

Figure 3. Real and imaginary branches of the dispersion relation for $h_s/h = \frac{1}{2}$, $\rho_s/\rho = 2$, $E_s/E = 4$ and $\nu = \nu_s = 0.3$. Solid lines are exact dispersion curves. In (*a*,*b*), the dotted lines are the Tiersten approximation, and in (*c*,*d*), they are the (Rayleigh–Lamb, finite-product) approximation with $(m_s, n_s) = (1, 1)$. (Online version in colour.)

(c) Extensions

A simple way to extend the range of validity of Tiersten's dispersion relation is to replace the approximate terms, such as $\tilde{D}_{\mathrm{Bs}}$ and $\tilde{D}_{\mathrm{Ss}}$, by their finite-product counterparts $D_{\mathrm{Bs}m_sn_s}$ and $D_{\mathrm{Ss}m_sn_s}$, in the notation of §5*b*. Other terms in $\tilde{Q}$ may be extended via finite-product approximations to the skin quantities in expressions (3.10)–(3.18) for $I_{\mathrm{C}}, I_{\mathrm{S}}$ and Q. Similarly, $\tilde{P}$ may be extended via $D_{\mathrm{BSs}m_sn_s}$ using expressions (3.10) and (3.13) for D_{BSs} and P. The result is a (Rayleigh–Lamb, finite-product) approximation, as elasticity theory is still being used for the core. Experimentation with values of (m_s, n_s), possibly with different choices made in different expressions, can be used to provide any desired enhancement of accuracy. Excellent results are obtained with $n_s = m_s$ or $n_s = m_s + 1$, from which unlimited accuracy is obtained on steadily increasing m_s.

(d) Numerical results

For the sandwich plate that we have been considering, with $h_s/h = \frac{1}{10}$, $E_s/E = 100$ and $\rho_s/\rho = 10$, the accuracy of Tiersten's method is comparable to that of the $(0, 1, 1, 2)$ approximation shown in figure 2*c*,*d*. This high accuracy is to be expected: because Tiersten's method uses the exact linear equations for the substrate, it always gives excellent results if the surface layer is thin, stiff and dense enough.

For a more demanding test of Tiersten's method, we take a thicker, softer, lighter surface layer, with $h_s/h = \frac{1}{2}$, $E_s/E = 4$, and $\rho_s/\rho = 2$. Tiersten's method now loses accuracy quite rapidly as the frequency increases, and is useful for values of Ω up to about 2 at most, as shown in figure 3*a*,*b*. By contrast, the (Rayleigh–Lamb, finite-product) approximation with (m_s, n_s) as low as $(1, 1)$ maintains accuracy up to much higher frequencies, beyond $\Omega = 6$, as shown in figure 3*c*,*d*. This is illustrates the scope provided by the finite-product method for finding the simplest approximation that gives any specified level of accuracy.

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7. Asymptotic decoupling and the composite Timoshenko approximation

When the elastic layers are all thin, a composite Timoshenko approximation may be derived, based on the transverse displacement $v(x,t)$ and shear angle $\theta(x,t)$ for the layers as a whole [12–15]. We now show that the results of this method may be written in asymptotically decoupled form, and we determine its numerical accuracy.

(a) The three-layered waveguide

For the three-layered waveguide specified in §2, composite Timoshenko equations for θ and v are

$$\rho_s h h_s(\theta_{tt} - 12c_{Bs}^2\theta_{xx}) + 2\rho c_2^2(\theta + v_x) = 0 \tag{7.1}$$

and

$$(\rho h + 2\rho_s h_s)hv_{tt} + (\rho c_B^2 h^3 + 2\rho_s c_{Bs}^2 h_s^3)hv_{xxxx} = \frac{1}{12}(\rho h^3 + 2\rho_s h_s^3)hv_{xxtt} + \rho c_2^2(h + h_s)^2(\theta_x + v_{xx}). \tag{7.2}$$

Solutions with θ and v proportional to $e^{-i\omega t + ikx}$ give a 2×2 determinant in ω and k whose value must be zero. This yields the composite Timoshenko dispersion relation.

(b) Asymptotic decoupling

The dispersion relation just defined is of the form

$$(\rho c_B^2 h_s)^2\hat{P} + (\rho c_B^2 h_s)(\rho_s c_{Bs}^2\, h)\hat{Q} + (\rho_s c_{Bs}^2\, h)^2\hat{R} = 0, \tag{7.3}$$

where $\hat{P}, \hat{Q}$ and $\hat{R}$ are Timoshenko analogues of the functions P, Q and R introduced in equation (3.1), and (c_B^2, c_{Bs}^2) replace the earlier (c_2^2, c_{2s}^2). Evaluation of the determinant leading to equation (7.3) gives

$$\hat{P} = \hat{D}_B, \quad \hat{R} = -\frac{1}{6(1-\nu)}\hat{D}_{Bs}\hat{D}_{Ss} \tag{7.4}$$

and

$$\hat{Q} = 2\Omega_{Bs}^2 + \frac{1}{6}(3K^2 + 6KK_s + 4K_s^2)\hat{D}_{Ss} - \frac{1}{12(1-\nu)}\hat{D}_{Bs}\hat{D}_{Ss}, \tag{7.5}$$

where

$$\hat{D}_B = \Omega_B^2 - K^4 + \tfrac{1}{4}\Omega_2^2 K^2, \quad \hat{D}_{Bs} = \Omega_{Bs}^2 - K_s^4 + \tfrac{1}{4}\Omega_{2s}^2 K_s^2 \quad \text{and} \quad \hat{D}_{Ss} = \frac{1}{\kappa}\Omega_{Bs}^2 - 12K_s^2. \tag{7.6}$$

Here, $\hat{D}_B$ and $\hat{D}_{Bs}$ include the Rayleigh correction term to the Euler–Bernoulli bending equation [25, p. 258], and $\hat{D}_{Ss}$ includes a correction factor κ which we shall take to be $\pi^2/12$. In making comparisons with the exact dispersion relation, a factor proportional to ω^6 needs to be cancelled from the approximations to the exact expressions P, Q, R to arrive at $\hat{P}, \hat{Q}, \hat{R}$. The limit of (7.3)–(7.6) at low frequency and wavenumber is the composite Euler–Bernoulli approximation (4.5). With some further terms retained, the limit may be written in the Timoshenko form

$$\Omega_{Bs}^2 = \frac{\rho_s h_s}{2\rho h}\left(\frac{1}{\kappa}\Omega_{Bs}^2 - 12K_s^2\right)\left\{\Omega_2^2 - \left(\frac{1 + 2h_s/h + (4/3)(h_s/h)^2}{1 + 2\rho_s h_s/(\rho h)}\right)K^2\right\}. \tag{7.7}$$

(c) Numerical results

For the stiff sandwich plate that we have been considering, with $h_s/h = \frac{1}{10}$, $E_s/E = 100$ and $\rho_s/\rho = 10$, the composite Timoshenko approximation has similar accuracy to the $(0,0,0,0)$ finite-product approximation with a correction factor $\kappa = \pi^2/12$, or to the $(0,0,0,1)$ finite-product approximation without a correction factor. This may be seen by noting the remarks at the end of §5*d* regarding the $(0,0,0,0)$ approximation, and by comparing figure 4 with figure 2*a*,*b*. Approximation (7.3)–(7.6) is plotted in figure 4*a*,*b*, and its reduced form (7.7) in figure 4*c*,*d*. The upper imaginary branch in

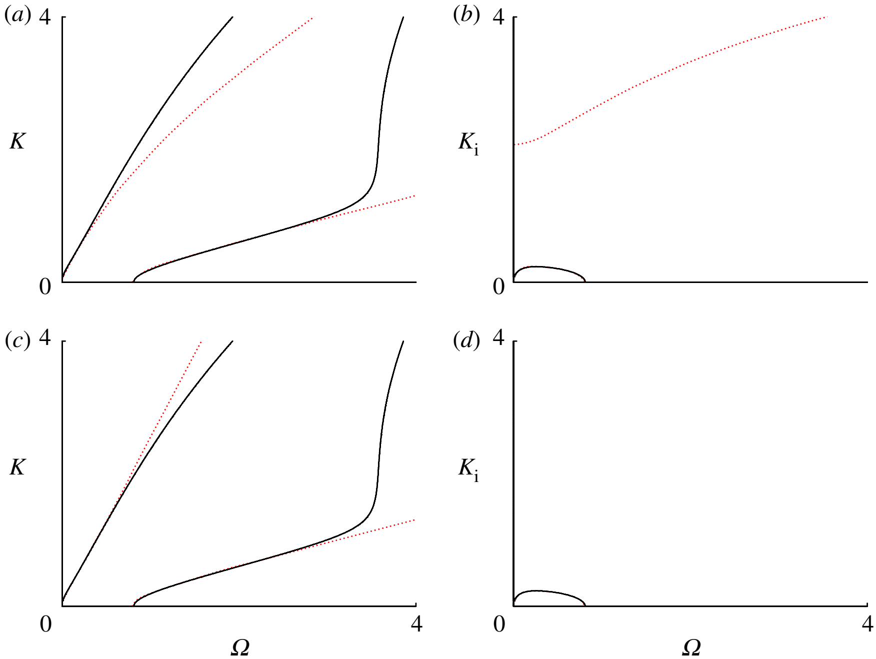

Figure 4. Real and imaginary branches of the dispersion relation for $h_s/h = \frac{1}{10}$, $\rho_s/\rho = 10$, $E_s/E = 100$ and $\nu = \nu_s = 0.3$. Solid lines are exact dispersion curves. In (*a*,*b*), the dotted lines are the composite Timoshenko approximation (7.3)–(7.6) with $\kappa = \pi^2/12$, and in (*c*,*d*), they are its reduced form (7.7). (Online version in colour.)

figure 4*b* is not given accurately by approximation (7.3)–(7.6); its accurate position is far higher, above the plotted region. Equation (7.7) and figure 4*c* agree with Liu & Bhattacharya [4].

8. Conclusions

This paper shows that asymptotic decoupling makes it possible to simplify and extend the range of useful polynomial approximations to dispersion relations for waves in layered media. The approach not only provides new and accurate results, but also shows that previous analyses, for example those based on the Tiersten thin-skin approximation or the composite Timoshenko equations, may be placed within a common hierarchical framework of clearly identified sub-problems. The mathematical basis of the paper, i.e. asymptotic factorization of a dispersion relation in certain limits, may be applied in many wave problems where wave interactions are important. The paper also demonstrates the usefulness of finite products as a tool in complex problems.

A by-product of the search for approximations is that a new analytical result has come to light, namely the explicit dispersion relation (3.1), with the given tractable expressions for P, Q and R, presented in a symmetrical and manageable form. An attractive feature of this analytical result is that its derivation and interpretation can be understood by physical reasoning about waves.

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