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The deferred limit method for long waves in a curved waveguide

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This paper presents a technique, based on a deferred approach to a limit, for analysing the dispersion relation for propagation of long waves in a curved waveguide. The technique involves the concept of an analytically satisfactory pair of Bessel functions, which is different from the concept of a numerically satisfactory pair, and simplifies the dispersion relations for curved waveguide problems. Details are presented for long elastic waves in a curved layer, for which symmetric and antisymmetric waves are strongly coupled. The technique gives high-order corrections to a widely used approximate dispersion relation based a kinematic hypothesis, and determines rigorously which of its coefficients are exact.

1. Introduction

Many approximate theories of long-wave propagation in circularly curved waveguides have been presented, based on kinematic assumptions about the shape of the field. For example, in an elastic layer, the most common assumption is that plane sections remain plane. Yet there is an omission regarding the scope of these theories. Although exact solutions of the governing equations have long been available [1], involving Bessel functions, the task of determining, mathematically, the limiting form of the corresponding dispersion relation as the wavelength increases has not been attempted. This task is worth performing for two reasons. The first is that it establishes the range of validity of any elementary method which starts from a kinematic assumption rather than the governing

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equations, and determines, for example, which Taylor series coefficients it finds correctly. The second reason is that it allows rational determination of correction factors which may be applied to an elementary formula in order to extend its range of validity, notably to higher frequencies, wavenumbers and layer thickness.

This paper provides the mathematical analysis which, starting from the exact linear equations for long waves propagating around a circular waveguide, leads to the polynomial dispersion relations of lowest order capturing the strong coupling of symmetric and antisymmetric waves, which is then the dominant feature of the field. Attention is paid to analytic structure involving fractional powers and logarithms, and to limits in which these do not occur. We also give higher order terms in the dispersion relation, which have not previously been calculated, and provide a method for calculating these terms to any order.

The regime of interest is that in which the curvature of the layer is great enough, or the wavelength long enough, that the field is not a small perturbation of the wave field in a flat layer. That is, the dispersion relation cannot be obtained by starting with decoupled equations for symmetric and anti-symmmetric waves in a flat layer, and adding a small coupling term. The asymptotic scaling which gives this regime of interest is established, and also its relation to the scalings available for regimes of lower curvature or smaller wavelength.

A mathematical concept arising in problems with cylindrical geometry is that of a numerically satisfactory pair of Bessel functions (J_K, Y_K). Such a pair is unsatisfactory here, because it introduces logarithms which ultimately cancel out, and it is better to use the analytically satisfactory pair (J_K, J_{-K}). This might be expected to fail at integer K, because the functions are then linearly dependent. However, it is shown that, by deferring the approach to the limit of integer K, the pair (J_K, J_{-K}) is superior to (J_K, Y_K) for all values of K in the dispersion relation. Although deferred limits arise in other contexts, for example in Richardson extrapolation [2, p. 375], the method of using an analytically satisfactory pair of Bessel functions and, where required, deferring an approach to a limit is believed to be new, and may be of use in a variety of problems with cylindrical geometry. Accordingly, the paper emphasizes those parts of the method likely to be of general interest. Although only L'Hôpital's rule is required, the type of formulae obtained are different from those currently found in the literature on waveguide theory. The fact that no logarithms appear in the final dispersion relation is a non-trivial result (here obtained rigorously), given the cylindrical geometry of the problem, and is a general result for curved waveguides.

An important aspect of this paper is comparison of rigorously derived approximations with those obtained from kinematic hypotheses. For elastic waveguides, the latter have a distinguished history [3–5] and have remained in widespread use [6–9], often with highly specific types deformation included [10,11]. A feature of any approach based on a kinematic hypothesis is that it contains no means within itself for determining the outer limits of its range of validity; comparison with an exact result is required [12]. We show that in certain respects, notably involving the width of the waveguide in relation to its radius of curvature, the range is greater than might be expected. Our results show that many research papers are too pessimistic in this respect.

The structure of the paper is as follows. Section 2 gives the required theory for the curved waveguide to be considered, using the Bessel functions (J_K, J_{-K}) throughout. Section 3 analyses the resulting dispersion relation, with attention to its symmetry properties and to how it may be reduced to a form with simple analytical structure, free of fractional powers or any possibility of a logarithmic limit. Section 4 shows how this form of dispersion relation gives not only the classical approximation for a thin curved layer, but also a family of correction terms to it. Section 5 presents a new type of approximate theory, in which no restriction is placed on the thickness of the layer; this theory gives an expansion of the dispersion relation in powers of frequency in which the coefficients are exact formulae in the thickness and wavenumber. Section 6 expands on some mathematical aspects of the method we have adopted; and §7 presents conclusions, including some directions for further work in waveguide theory.

2. Elastic waves in a curved layer

(a) Governing equations

An isotropic elastic medium occupies a curved layer of width h represented in cylindrical polar coordinates (r,θ,z) by $a_\mathrm{i}<r<a_\mathrm{e}$ and $\theta_-<\theta<\theta_+$, where $a_\mathrm{i}=a-h/2$ and $a_\mathrm{e}=a+h/2$. Here subscripts (i, e) indicate the (interior, exterior) boundaries of the layer, at which traction-free boundary-conditions are applied. The dimensionless width is $\epsilon=h/a$, which is equivalently a measure of the curvature of the layer (figure 1).

The elastic medium supports wave motion satisfying Navier's equations [5, p. 274]

$$\frac{\partial^2\mathbf{u}}{\partial t^2}=(c_1^2-c_2^2)\nabla\nabla\cdot\mathbf{u}+c_2^2\nabla^2\mathbf{u}, \tag{2.1}$$

where $\mathbf{u}$ is the displacement, with components (u,v,w) in cylindrical polar coordinates. A subscript 1 refers to P-waves, and a subscript 2 refers to S-waves; thus c_1 is the P-wave speed, i.e. compression-wave speed, and c_2 is the S-wave speed, i.e. shear-wave speed. The medium has Young's modulus E, Poisson's ratio ν, density ρ and reference speed $c_0=(E/\rho)^{1/2}$. In plane strain, the wave speeds are

$$c_1=\left\{\frac{1-\nu}{(1+\nu)(1-2\nu)}\right\}^{1/2}c_0,\quad c_2=\frac{c_0}{\{2(1+\nu)\}^{1/2}}. \tag{2.2}$$

For plane stress, the value of c_1 is replaced by $c_0/\{(1-\nu^2)\}^{1/2}$. Another reference speed is the bending-wave characteristic speed c_B, defined by $c_\mathrm{B}^2=c_2^2(c_1^2-c_2^2)/(3c_1^2)$. For wave motion at frequency ω, we use dimensionless frequencies $\Omega=\omega a/c_0$ and $\Omega_\mathrm{B}=\omega a/c_\mathrm{B}$ scaled on the radius, or $\Omega_\mathrm{s}=\omega h/c_0$ and $\Omega_\mathrm{Bs}=\omega h/c_\mathrm{B}$ scaled on the thickness. Here, the subscript s stands for skin.

Two-dimensional wave motion with $w=0$ has the Lamé decomposition [5, p. 275]

$$\mathbf{u}=\nabla\phi+\nabla\times(\psi\mathbf{e}_z), \tag{2.3}$$

where $\phi=\phi(r,\theta,t)$, $\psi=\psi(r,\theta,t)$ and $\mathbf{e}_z$ is the unit coordinate vector in the z direction. The Lamé potentials ϕ and ψ satisfy the wave equations

$$\frac{\partial^2\phi}{\partial t^2}=c_1^2\nabla^2\phi,\quad \frac{\partial^2\psi}{\partial t^2}=c_2^2\nabla^2\psi, \tag{2.4}$$

and are coupled only through the boundary conditions. These two-dimensional equations are appropriate for a curved layer in either plane strain or plane stress, with the appropriate values of c_1 and c_2, as given above.

The strain components are

$$e_{rr}=\frac{\partial u}{\partial r},\quad e_{\theta\theta}=\frac{1}{r}\left(\frac{\partial v}{\partial\theta}+u\right),\quad e_{r\theta}=\frac{1}{2}\left\{r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{r}\frac{\partial u}{\partial\theta}\right\}, \tag{2.5}$$

and the required stress components are

$$\tau_{rr}=\rho\{c_1^2e_{rr}+(c_1^2-2c_2^2)e_{\theta\theta}\},\quad \tau_{r\theta}=2\rho c_2^2e_{r\theta}. \tag{2.6}$$

Traction-free boundary conditions are $\tau_{rr}=0$ and $\tau_{r\theta}=0$.

(b) Wave propagation

We seek solutions of the governing equations with real frequency ω and circumferential wavenumber k, which may be complex, in which all components are proportional to $\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}+\mathrm{i}\mathrm{k}\mathrm{a}\theta}$. This factor is omitted from all field expressions, so that the part written explicitly is a function of

r only. Thus, we write the Lamé potentials in the form

$$\phi = \mathrm{i}\Phi(r), \quad \psi = \Psi(r), \tag{2.7}$$

so that

$$\nabla\phi = \left(\mathrm{i}\Phi_r, -\frac{K}{r}\Phi\right), \quad \nabla \times (\psi \mathbf{e}_z) = \left(\frac{\mathrm{iK}}{r}\Psi, -\Psi_r\right), \tag{2.8}$$

where $K = ka$ is the dimensionless wavenumber measured along the centre-line of the layer, and the subscript r denotes differentiation with respect to r. From (2.3), the displacement components are

$$(u, v) = \left(\mathrm{i}\Phi_r + \frac{\mathrm{iK}}{r}\Psi, -\frac{K}{r}\Phi - \Psi_r\right), \tag{2.9}$$

from which the strain and stress components given in (2.5) and (2.6) are readily found by differentiation. We omit z components, which are zero because the displacement is two-dimensional with $w = 0$. The wave equations (2.4) give Bessel's equation in the two forms

$$r^2\Phi_{rr} + r\Phi_r + \left\{\left(\frac{\omega r}{c_1}\right)^2 - K^2\right\}\Phi = 0 \tag{2.10}$$

and

$$r^2\Psi_{rr} + r\Psi_r + \left\{\left(\frac{\omega r}{c_2}\right)^2 - K^2\right\}\Psi = 0, \tag{2.11}$$

and the boundary conditions $\tau_{rr} = 0$ and $\tau_{r\theta} = 0$ become

$$r\Phi_r + \left\{\frac{1}{2}\left(\frac{\omega r}{c_2}\right)^2 - K^2\right\}\Phi = K(r\Psi_r - \Psi) \tag{2.12}$$

and

$$r\Psi_r + \left\{\frac{1}{2}\left(\frac{\omega r}{c_2}\right)^2 - K^2\right\}\Psi = K(r\Phi_r - \Phi). \tag{2.13}$$

(c) Analytically satisfactory solutions

A standard topic in the theory of differential equations is the idea of a numerically satisfactory pair of solutions ([13, p. 154], [14, p. 218]), chosen to avoid linear dependence, or severe cancellation near singular points. For Bessel's equation (2.10)–(2.11) and the type of boundary-value problem we are considering, a numerically satisfactory pair is (J_K, Y_K), and this is the choice made by previous writers on the subject of waves in a curved layer, e.g. [1,15].

In the present investigation, the above choice is unsatisfactory from the analytical point of view. In particular, the use of Y_K as the second solution has two disadvantages. The first is that at integer values of K the series expansion of Y_K involves a logarithm—yet there is no logarithmic term, either approximately or exactly, in the dispersion relation: all the logarithms introduced by Y_K ultimately cancel out. The second disadvantage of Y_K is that the underlying problem is symmetric in K, in that the transformation $K \to -K$ sends the waves in the opposite direction around the curved layer, but introduces no other change to the problem. This symmetry is hidden by the use of Y_K; for example, until a late stage in the calculation, there is little indication that the final dispersion relation will involve only even powers of K, as it must.

The above disadvantages may be overcome by using row operations on a determinant, in conjunction with the identities

$$J_{-K}(\Omega) = \cos(\pi K)\, J_K(\Omega) - \sin(\pi K)\, Y_K(\Omega) \tag{2.14}$$

and

$$J_{K+1}(\Omega) = \frac{2K}{\Omega} J_K(\Omega) - J_{K-1}(\Omega). \tag{2.15}$$

However, when this is done, it becomes apparent that (J_K, J_{-K}) could have been used from the outset as an 'analytically satisfactory' pair of solutions, even though, because of their linear

dependence for integer K, they are not numerically satisfactory. It is merely necessary to defer any limiting operation. For example, if the limit $K \to 0$ is deferred to the end of the calculation, all difficulties vanish, because of cancellation of powers such as Ω^K and Ω^{-K} in products. By contrast, the use of Y_K at the outset already involves a limit, because the standard definition of Y_K takes a degenerate form involving a zero numerator and denominator when K is an integer, and Y_K must then be defined as a limit by L'Hôpital's rule. Thus, there must be a limit somewhere, and it is simply a case of whether one takes this limit at the beginning of the calculation, or defers it to the end. The deferred limit method maintains the symmetry between K and $-K$ at every stage of the work. As this symmetry is present in most waveguide problems, our approach is consistent with the aim of always exploiting the symmetries in a problem to the full.

Thus, we write the general solution of (2.10)–(2.11) in the analytically satisfactory form

$$\Phi = A_1^+ J_{K,1} + A_1^- J_{-K,1} \tag{2.16}$$

and

$$\Psi = A_2^+ J_{K,2} - A_2^- J_{-K,2}. \tag{2.17}$$

Here, $A_1^\pm$ and $A_2^\pm$ are modal coefficients, and the subscripts 1 and 2 attached to the Bessel functions denote evaluation at arguments $\omega r/c_1$ and $\omega r/c_2$; thus $J_{-K,2}$ indicates $J_{-K}(\omega r/c_2)$, for example. The signs before A_1^- and A_2^- preserve the symmetry of later formulae; this choice is helpful because the disposition of the grad and curl operators in the Lamé decomposition (2.3) suggests that we should make Φ symmetric and Ψ antisymmetric with regard to change of superscripted signs in (2.16)–(2.17), consistent with our philosophy of exploiting symmetry.

(d) Boundary conditions

With Φ and Ψ as above, the boundary conditions (2.12)–(2.13) applied at $r = a_\mathrm{i} = a - h/2$ and $r = a_\mathrm{e} = a + h/2$ may be written as

$$A_1^+ f_{1\mathrm{i}}^+ + A_1^- f_{1\mathrm{i}}^- = A_2^+ g_{2\mathrm{i}}^+ + A_2^- g_{2\mathrm{i}}^-, \tag{2.18}$$

$$A_1^+ g_{1\mathrm{i}}^+ - A_1^- g_{1\mathrm{i}}^- = A_2^+ f_{2\mathrm{i}}^+ - A_2^- f_{2\mathrm{i}}^-, \tag{2.19}$$

$$A_1^+ f_{1\mathrm{e}}^+ + A_1^- f_{1\mathrm{e}}^- = A_2^+ g_{2\mathrm{e}}^+ + A_2^- g_{2\mathrm{e}}^- \tag{2.20}$$

and

$$A_1^+ g_{1\mathrm{e}}^+ - A_1^- g_{1\mathrm{e}}^- = A_2^+ f_{2\mathrm{e}}^+ - A_2^- f_{2\mathrm{e}}^-. \tag{2.21}$$

Here we make use of the functions

$$f(\Omega, K) = \left\{ \tfrac{1}{2}\Omega_2^2 - K(K-1) \right\} J_K(\Omega) - \Omega J_{K+1}(\Omega) \tag{2.22}$$

and

$$g(\Omega, K) = K(K-1) J_K(\Omega) - \Omega K J_{K+1}(\Omega) \tag{2.23}$$

evaluated at the dimensionless frequencies $\Omega = \Omega_{1\mathrm{i}}, \Omega_{1\mathrm{e}}, \ldots$ defined by

$$\Omega_{1\mathrm{i}} = \frac{\omega a_\mathrm{i}}{c_1}, \quad \Omega_{1\mathrm{e}} = \frac{\omega a_\mathrm{e}}{c_1}, \quad \Omega_{2\mathrm{i}} = \frac{\omega a_\mathrm{i}}{c_2}, \quad \Omega_{2\mathrm{e}} = \frac{\omega a_\mathrm{e}}{c_2}, \tag{2.24}$$

and at the wavenumbers $\pm K$ as indicated by the subscripts and superscripts on f and g; for example,

$$f_{1\mathrm{i}}^+ = f(\Omega_{1\mathrm{i}}, K), \quad g_{2\mathrm{e}}^- = g(\Omega_{2\mathrm{e}}, -K). \tag{2.25}$$

In the definition (2.22), the quantity $\frac{1}{2}\Omega_2^2$ is not an argument of the function f; it is evaluated at $\Omega_{2\mathrm{i}}$ or $\Omega_{2\mathrm{e}}$, always with a subscript 2. This is a consequence of the way the term $\frac{1}{2}(\omega r/c_2)^2$ enters the boundary conditions (2.12)–(2.13); recall that the function Φ is associated with the wave speed c_1, and the function Ψ is associated with the wave speed c_2.

3. The dispersion relation

(a) Symmetry of the basic determinant

For a non-trivial solution of the governing equations, the determinant of the simultaneous equations (2.18)–(2.21) for the modal coefficients $A_1^{\pm}$ and $A_2^{\pm}$ must be zero, i.e.

$$\begin{vmatrix} f_{1\mathrm{i}}^{+} & f_{1\mathrm{i}}^{-} & g_{2\mathrm{i}}^{+} & g_{2\mathrm{i}}^{-} \\ g_{1\mathrm{i}}^{+} & -g_{1\mathrm{i}}^{-} & f_{2\mathrm{i}}^{+} & -f_{2\mathrm{i}}^{-} \\ f_{1\mathrm{e}}^{+} & f_{1\mathrm{e}}^{-} & g_{2\mathrm{e}}^{+} & g_{2\mathrm{e}}^{-} \\ g_{1\mathrm{e}}^{+} & -g_{1\mathrm{e}}^{-} & f_{2\mathrm{e}}^{+} & -f_{2\mathrm{e}}^{-} \end{vmatrix} = 0. \tag{3.1}$$

Regarded as a relation between ω and k, this is the dispersion relation for elastic waves in a curved layer. It is an exact result involving two independent functions, f and g, which represents symmetries of the wave motion by transpositions of (f,g), $(1,2)$, $(+,-)$ and (i,e). The $(1,2)$ symmetry, except for the term $\frac{1}{2}\Omega_2^2$ in the definition of f, is notable, because P-waves and S-waves are not interchangeable.

(b) Analytic structure of the dispersion relation

The Bessel function J_K is defined by

$$J_K(\Omega) = \frac{((1/2)\Omega)^K}{K!} \sum_{s=0}^{\infty} \frac{(-(1/4)\Omega^2)^s}{s!(1+K)_s}, \tag{3.2}$$

in which we use the Pochhammer symbol defined by $(x)_0 = 1$ and $(x)_s = x(x+1)\cdots(x+s-1)$ for $s = 1, 2, \ldots$. The pre-factor $(\frac{1}{2}\Omega)^K/K!$ is transcendental, but the terms after the summation sign in (3.2) are rational. Therefore, in a product of Bessel functions $J_{\pm K}$, possibly with different arguments, much of the transcendental behaviour in the prefactors cancels out. After truncation at finite order, the result is a rational function.

The same remarks apply to the functions f and g defined in (2.22)–(2.23). These may be written as

$$f(\Omega,K) = \frac{((1/2)\Omega)^K}{K!} F(\Omega,K), \quad g(\Omega,K) = \frac{((1/2)\Omega)^K}{K!} G(\Omega,K), \tag{3.3}$$

where

$$F(\Omega,K) = \sum_{s=0}^{\infty} \left\{ \frac{1}{2}\Omega_2^2 - K(K-1) + 2s \right\} \frac{(-(1/4)\Omega^2)^s}{s!(1+K)_s} \tag{3.4}$$

and

$$G(\Omega,K) = \sum_{s=0}^{\infty} \{K(K-1) + 2Ks\} \frac{(-(1/4)\Omega^2)^s}{s!(1+K)_s}. \tag{3.5}$$

Thus in a product involving f and g, most of the prefactor cancels, to leave a simpler expression in F and G. This occurs in the determinant (3.1): the terms in $(\frac{1}{2}\Omega_{1\mathrm{i}})^{\pm K}$, $(\frac{1}{2}\Omega_{1\mathrm{e}})^{\pm K}$, ... cancel out in every product, except for a residual part involving $a_{\mathrm{i}}^{\pm K}$ and $a_{\mathrm{e}}^{\pm K}$, to leave in the factor only this residual part and the term

$$\frac{1}{\{K!(-K)!\}^2} = \left(\frac{\sin \pi K}{\pi K} \right)^2. \tag{3.6}$$

This too cancels out, being the same in every product. Hence, the dispersion relation (3.1) may be replaced by

$$\begin{vmatrix} F_{1\mathrm{i}}^{+} & F_{1\mathrm{i}}^{-} & G_{2\mathrm{i}}^{+} & G_{2\mathrm{i}}^{-} \\ G_{1\mathrm{i}}^{+} & -G_{1\mathrm{i}}^{-} & F_{2\mathrm{i}}^{+} & -F_{2\mathrm{i}}^{-} \\ F_{1\mathrm{e}}^{+} & F_{1\mathrm{e}}^{-} & G_{2\mathrm{e}}^{+} & G_{2\mathrm{e}}^{-} \\ G_{1\mathrm{e}}^{+} & -G_{1\mathrm{e}}^{-} & F_{2\mathrm{e}}^{+} & -F_{2\mathrm{e}}^{-} \end{vmatrix} = 0, \tag{3.7}$$

in which the notation corresponds to that in (2.25) but with a factor $a_{\mathrm{i}}^{\pm K}$ or $a_{\mathrm{e}}^{\pm K}$; for example,

$$F_{1\mathrm{i}}^{+} = a_{\mathrm{i}}^{K} F(\Omega_{1\mathrm{i}}, K), \quad G_{2\mathrm{e}}^{-} = a_{\mathrm{e}}^{-K} G(\Omega_{2\mathrm{e}}, -K). \tag{3.8}$$

Dimensionless factors may be used instead, for example $(a_{\mathrm{i}}/a)^{\pm K}$ and $(a_{\mathrm{e}}/a)^{\pm K}$ based on the arithmetic mean a of the radii. Thus the factors may be written as $(1 \pm \frac{1}{2}\epsilon)^{\pm K}$, in which all combinations of signs are required. In a product of four terms, the factor is $(a_{\mathrm{e}}/a_{\mathrm{i}})^{\pm 2K}$, that is $\{(1 + \frac{1}{2}\epsilon)/(1 - \frac{1}{2}\epsilon)\}^{\pm 2K}$.

The key analytical property of the determinant (3.7) is that each of its terms has a Taylor series about the point $(\omega, k) = (0, 0)$. As indicated in §2c, there is no logarithm, either in the exact expression for the dispersion relation, or in its approximations obtained by truncating the infinite series. The coefficients in the series (3.4) and (3.5) contain only rational expressions. In the early terms of the Taylor series expansion of the dispersion relation, the integers in the resulting numerators and denominators are quite small.

Removal of pre-factors occurs also in the definition of Bessel–Clifford functions [16]. However, we go further, in removing both a power and a factorial. As will be seen in §6a, this is necessary for the success of the method, because (3.6) enters the later mathematical theory.

(c) Symmetric groupings of terms

The terms obtained from (3.1) may be separated into six groups of four terms each, in such a way that each group is symmetric under the eight symmetries obtained by independently transposing or leaving alone the elements of pairs (F, G), $(+, -)$, and (i, e); we use (F, G) rather than (f, g). For example, two of the six groups are

$$\begin{pmatrix} F_{1\mathrm{i}}^{+} G_{1\mathrm{i}}^{-} F_{2\mathrm{e}}^{-} G_{2\mathrm{e}}^{+} & F_{1\mathrm{i}}^{-} G_{1\mathrm{i}}^{+} F_{2\mathrm{e}}^{+} G_{2\mathrm{e}}^{-} \\ F_{1\mathrm{e}}^{+} G_{1\mathrm{e}}^{-} F_{2\mathrm{i}}^{-} G_{2\mathrm{i}}^{+} & F_{1\mathrm{e}}^{-} G_{1\mathrm{e}}^{+} F_{2\mathrm{i}}^{+} G_{2\mathrm{i}}^{-} \end{pmatrix}, \quad \begin{pmatrix} -F_{1\mathrm{i}}^{+} F_{2\mathrm{i}}^{-} G_{1\mathrm{e}}^{-} G_{2\mathrm{e}}^{+} & -F_{1\mathrm{i}}^{-} F_{2\mathrm{i}}^{+} G_{1\mathrm{e}}^{+} G_{2\mathrm{e}}^{-} \\ -F_{1\mathrm{e}}^{+} F_{2\mathrm{e}}^{-} G_{1\mathrm{i}}^{-} G_{2\mathrm{i}}^{+} & -F_{1\mathrm{e}}^{-} F_{2\mathrm{e}}^{+} G_{1\mathrm{i}}^{+} G_{2\mathrm{i}}^{-} \end{pmatrix}, \tag{3.9}$$

in which the terms in each group are to be added. Exchanging (i, e) permutes the rows; and exchanging (F, G) or $(+, -)$ permutes the terms within each group in less obvious ways. There are no symmetries under transposition of $(1, 2)$, as is obvious on physical grounds, because *P*-waves and *S*-waves are not interchangeable. The six sums of four terms are basic objects in the dispersion relation, because smaller groupings could not be physically meaningful, given the symmetries in the problem.

(d) Plots of the dispersion relation

Plots of the first few branches of the dispersion relation (3.1) are given as solid curves in figure 2. The lower part of the first two branches is shown in more detail in figure 3. In what follows, we first show how the standard approximations in this region follow from the exact dispersion relation written in the form (3.7). We then derive new results from (3.7), and relate them to the plots. To organize our work, we present first an elementary class of results, valid for a thin layer, and then a more advanced class, valid for arbitrary thickness. The results do not require the dimensionless wavenumber K to be small. Nevertheless, we determine the limiting forms of certain expressions when $K \ll 1$, and present some explicit results for $K = 0$, including new approximations for the first cut-on frequency, known as the ring frequency.

4. Elementary theory for a thin layer

(a) Leading order terms

Our approach is to expand the dispersion relation (3.7) in powers of the frequency, and then expand each coefficient separately in powers of $\epsilon = h/a$. The terms have a common factor, corresponding to translations and rotations of the elastic layer, which are not of interest here.

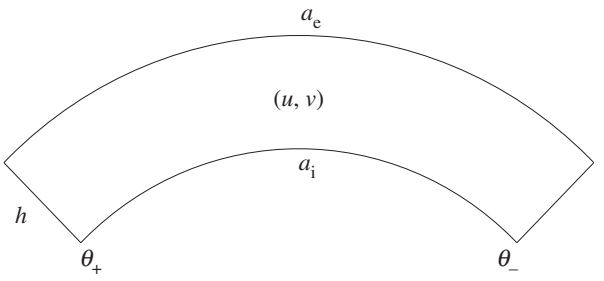

Figure 1. A curved elastic layer occupying the region $a_i < r < a_e$ and $\theta_- < \theta < \theta_+$. The thickness of the layer is $h = a_e - a_i$, and the (r, θ) displacement components are (u, v).

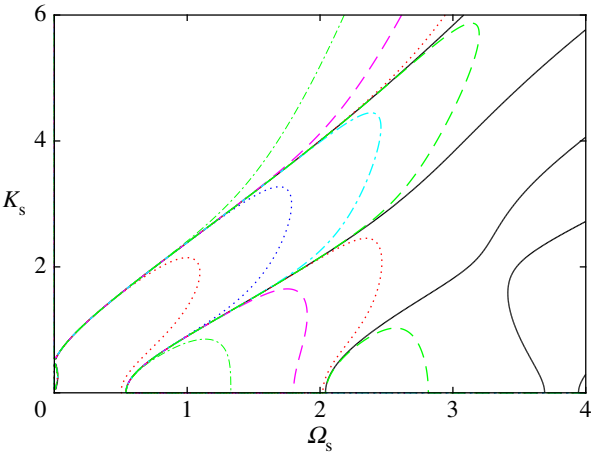

Figure 2. Exact and approximate dispersion relation for a curved layer in plane strain with Poisson's ratio $\nu = 0.3$ and dimensionless width $\epsilon = h/a = \frac{1}{2}$. The axes are the thickness-scaled frequency $\Omega_s = \omega h/c_0$ and thickness-scaled wavenumber $K_s = ka$. Exact curves are solid (black); approximate curves, obtained from truncations of (5.3) to orders $\tilde{\Omega}^4$, $\tilde{\Omega}^6$, . . ., up to order $\tilde{\Omega}^{16}$, are dashed (red), dash-dotted (green), dotted (blue), . . ., progressively extending the range of accuracy to higher frequencies and wavenumbers. In the small arc at the bottom left, the curves are indistinguishable on the scale of the graph.

It is necessary to divide throughout by this factor, which is of the form $\epsilon^2 K^2 (1 - K^2)^2 \Omega^4$. The leading terms of (3.7) are then found to be

$$\epsilon^2 K^2 (1 - K^2)^2 - (1 + K^2)\Omega_B^2 + \tfrac{1}{12}\Omega_B^4 = 0. \tag{4.1}$$

This equation captures the dominant behaviour of the dispersion curves in the strongly coupled regime, and also in that part of the flat-plate regime represented by the Bernoulli approximation for bending waves, or its equivalent (the Poisson approximation) for stretching waves. It does this in the most economical way possible, in that if the dominant behaviour in these regimes is required, then no further terms should be included beyond those given in (4.1), because they would be small, and, conversely, every term given must be included. Although (4.1) is similar to various classical equations [3–5] which have been widely used [6–9,12], it is new in the precise choice of the terms included or excluded, as discussed further below.

Equation (4.1) is the first equation of its type to have been derived from the governing equations rather than from a kinematic hypothesis assumed at the outset. Numerical comparisons with the exact dispersion relation show that it is accurate up to about $\epsilon = \frac{1}{2}$, when the discrepancy

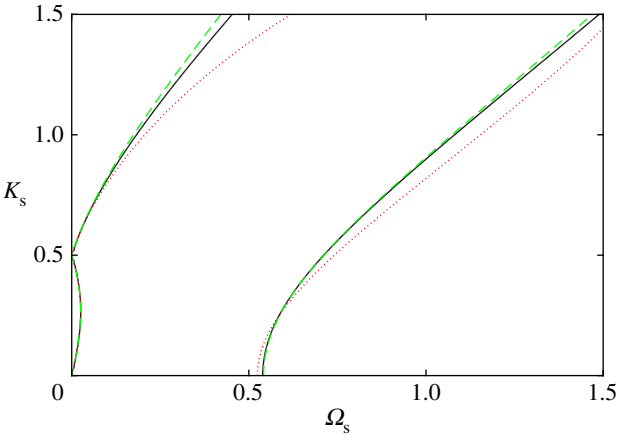

Figure 3. Exact and approximate dispersion relation for a curved layer in plane strain with Poisson's ratio $\nu = 0.3$ and dimensionless width $\epsilon = h/a = \frac{1}{2}$. The solid (black) curves are exact; the dotted (red) curves are the elementary approximation (4.1); and the dashed (green) curves are obtained from the series (4.5) by retaining terms up to order ϵ^2 after making the substitutions (4.6)–(4.9). On the scale of the graph, the intersection of the dashed (green) curve with the frequency axis near $\Omega_s = 0.5$ is indistinguishable from the exact first cut-on frequency (i.e. the ring frequency) obtained from the solid (black) line; this approximation is equivalent to that obtained by retaining terms up to order ϵ^2 in (4.11).

in the cut-on frequency of the second branch (i.e. the ring frequency) starts to become noticeable, as indicated in figure 3.

(i) Subordinate regimes

The dispersion relation (4.1) has two subordinate regimes, corresponding to the two branches in figure 3, which arise from two dominant balances of terms. The first is the regime $K = O(1)$, $\Omega_B = O(\epsilon)$ or equivalently $K_s = O(\epsilon)$, $\Omega_{Bs} = O(\epsilon^2)$, obtained from the first two terms only; the second is the regime $K = O(1)$, $\Omega_B = O(1)$ or equivalently $K_s = O(\epsilon)$, $\Omega_{Bs} = O(\epsilon)$, obtained from the second and third terms only. These give the leading order equations

$$\Omega_B^2 = \epsilon^2 \frac{K^2(1-K^2)^2}{1+K^2}, \quad \frac{1}{12}\Omega_B^2 = 1 + K^2, \tag{4.2}$$

for the branches in figure 3. The first branch is the slow wave, with phase speed of order ϵ, i.e. the curvature-modified version of a bending wave; the second branch is the fast wave, with phase speed which varies from order 1 (at large K) to infinity (at $K = 0$) as the ring frequency $\Omega_B \simeq \sqrt{12}$ is approached. The full version (4.1) gives accurately the complex branches linking the fast and slow wave [12]. Two properties of the slow wave are that its Taylor series at the origin begins with the term $\Omega_B = \epsilon K$, and that at leading order the local maximum of the frequency is $\Omega_B = \{(71 - 17\sqrt{17})/8\}^{1/2}\epsilon = 0.3368\epsilon$, attained at $K = \{(\sqrt{17} - 3)/4\}^{1/2} = 0.5299$.

(ii) Comparison with shell theory

Our approach may be compared with the shallow shell theory of Donnell–Mushtari–Vlasov, as used in [12], for example. This theory gives, in our notation, the equation

$$\left\{\tfrac{1}{12}\Omega_B^2 - (1 + \tfrac{1}{12}\epsilon^2 K^4)\right\}\left\{\tfrac{1}{12}\Omega_B^2 - (1 + \tfrac{1}{12}\epsilon^2)K^2\right\} = K^2(1 + \tfrac{1}{12}\epsilon^2 K^2)^2, \tag{4.3}$$

in which the factors on the l.h.s. represent bending and stretching waves, as modified by the curvature of the layer, and the r.h.s. represents their interaction. On expanding this equation, the result is (4.1), but with a factor $1 + \frac{1}{12}\epsilon^2 K^2$ multiplying the central term. However, the term $\frac{1}{12}\epsilon^2 K^2$ here does not contribute to (4.1) at leading order when $K \ll 1/\epsilon$, and may be ignored, so that

(4.3) agrees with (4.1). As we have derived (4.1) from the exact equations of linear elasticity, our results establish rigorously the leading-order correctness of Donnell–Mushtari–Vlasov shallow shell theory for the curved layer we are analysing. It should be noted that no individual term in (4.3) may be ignored, because all three of its terms containing ϵ^2 contribute to the term $\epsilon^2K^2(1-K^2)^2$ in (4.1), and it is essential to keep this term without further approximation in order to be correct at leading order.

(iii) The flat-plate limit

We now use (4.1) to derive the flat-plate limit. A check reveals that in the regime $1 \ll K \ll 1/\epsilon$ and $\epsilon \ll \Omega \ll 1/\epsilon$, a dominant balance gives $\Omega_B^2 = \epsilon^2K^4$, which in thickness-scaled variables is $\Omega_{Bs}^2 = K_s^4$, i.e. the Bernoulli approximation for bending waves in a flat plate. Similarly, in the regime $1 \ll K \ll 1/\epsilon$ and $1 \ll \Omega \ll 1/\epsilon$, a dominant balance gives $\Omega_B^2 = 12\,K^2$, or equivalently $\Omega_{Bs}^2 = 12K_s^2$, i.e. the Poisson approximation for stretching waves. The condition $1 \ll K \ll 1/\epsilon$ is that the wavelength should be much less than the radius of curvature of the layer, but much larger than the thickness. Here the condition $K \ll 1/\epsilon$, relates to the validity of the Bernoulli and Poisson approximations to flat-plate waves, rather than the occurrence of flat-plate behaviour itself. In a thin curved layer, flat-plate behaviour occurs for all large enough K, but when K becomes of order $1/\epsilon$ or larger, the elementary approximations break down, and the full Rayleigh–Lamb equations for a flat plate are required. The two flat-plate limits taken together correspond in (4.1) to retaining only the terms

$$\epsilon^2K^6 - K^2\Omega_B^2 + \tfrac{1}{12}\Omega_B^4 = 0. \tag{4.4}$$

(b) Higher order terms

The complete series expansion of the dispersion relation (3.7) in powers of the thickness and frequency is readily obtained to any order with the aid of a short Mathematica program. It is convenient to express the results in terms of the radius-scaled variables $\Omega = \omega a/c_0$ and $K = ka$, and use the notation $\alpha = (c_1/c_0)^2$ and $\beta = (c_2/c_0)^2$. After division by the common factor $\epsilon^2K^2(1-K^2)^2\Omega^4$ the result may be written as

$$a_0\epsilon^2K^2(1-K^2)^2 + a_2\Omega^2 + (b_0 + b_2\epsilon^2\Omega^2 + b_4\epsilon^4\Omega^4 + \cdots)\Omega^4 = 0, \tag{4.5}$$

in which the coefficients a_0, a_2, b_0, b_2, ... are functions of K and ϵ with leading term of order one. The leading powers of ϵ in (4.5) are obtained either from the determinant (3.7), or from the elementary theory leading to (4.1). Expression (4.5) is exact if the series is continued to infinity. The first few coefficients are

$$a_0 = \frac{4(\alpha-\beta)^2}{3\alpha^2}\left\{1 - \frac{1}{30}(1-4K^2)\epsilon^2 - \frac{1}{1680}(1-4K^2)(5+4K^2)\epsilon^4 + \cdots\right\}, \tag{4.6}$$

$$a_2 = -\frac{4(\alpha-\beta)}{\alpha\beta}\left\{1 + K^2 + \frac{1}{12\alpha}\left[\alpha(3-5K^2+8K^4) + 4\beta K^2(1-K^2)\right]\epsilon^2 + \cdots\right\}, \tag{4.7}$$

$$b_0 = \frac{1}{\beta^2}\left\{1 + \frac{1}{6\alpha^2}[\alpha^2(3+8K^2) + 4\alpha\beta(1-K^2) - 2\beta^2(2+K^2)]\epsilon^2 + \cdots\right\}, \tag{4.8}$$

$$b_2 = -\frac{1}{6\alpha\beta^3}\left\{\alpha+\beta + \frac{1}{60\alpha^2}[\alpha^3(-13+36K^2) + \alpha^2\beta(59+36K^2) - 4\alpha\beta^2(1+9K^2) - 4\beta^3(3+K^2)]\epsilon^2 + \cdots\right\}, \tag{4.9}$$

and

$$b_4 = \frac{1}{360\alpha^2\beta^4}\{(\alpha+3\beta)(3\alpha+\beta) + \cdots\}. \tag{4.10}$$

Only polynomials in K occur here, despite the high-order denominators in the definitions (3.4) and (3.5) of F and G, and the products of such terms in the determinant (3.7); as we are not expanding in powers of K, the expressions in K in (4.6)–(4.10) are exact. We explain in §6a why

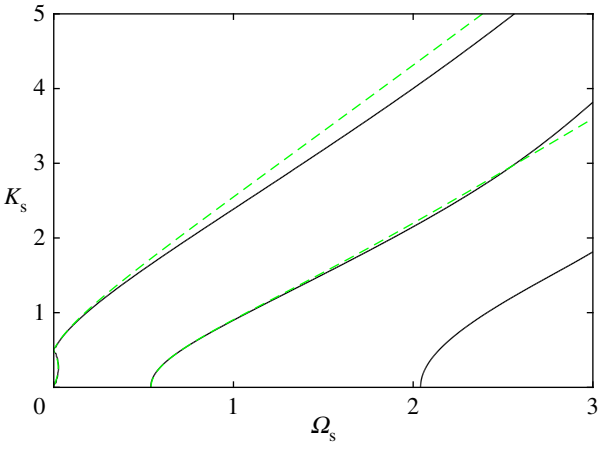

Figure 4. Extension of figure 3 to higher frequencies and wavenumbers, retaining only the exact dispersion relation, i.e. the solid (black) curves, and approximation (4.5) up to terms of order ϵ^2, i.e. the dashed (green) curves. The parameter values are the same, $\nu = 0.3$ and $\epsilon = \frac{1}{2}$. This approximation has a wide range of validity.

only polynomials in K can occur in the series expansion of the determinant (3.7) in powers of Ω; this makes poles in the K-plane impossible.

Expansion (4.5) suggests higher-order approximations. The most obvious correction to the elementary theory of §4a is to retain all terms up to order ϵ^2 in (4.5), after substituting the series expansions (4.6)–(4.10) for the coefficients. Thus, we approximate a_0 and b_2 by their first terms only, a_2 and b_0 by their first two terms, and omit all other terms. The result is plotted as the dashed curve in figure 3. The second branch is now captured to almost perfect accuracy, even though $\epsilon = \frac{1}{2}$. This value of ϵ corresponds to $a_e/a_i = 3$; that is, the outer radius of the layer is three times the inner radius. Figure 4 shows the range of frequencies and wavenumbers for which the approximation remains accurate. These figures demonstrate that with properly chosen terms included, thin-layer asymptotics for strongly coupled waves is accurate well beyond its expected range of validity.

(c) The small wavenumber limit

A special case of the formulae in §4b is $K = 0$, i.e. the frequency axis in the dispersion diagram. The first term in (4.5) then vanishes, and we may divide by Ω^2; the resulting equation gives the cut-on frequencies as a function of ϵ. In principle, one could obtain this equation by successive differentiation of the exact dispersion relation expressed in terms of Bessel functions; but as the first non-zero derivatives occur after a common factor which by now is $\epsilon^2K^2(1-K^2)^2\Omega^6$, this would be an impractical undertaking because so many of the initial derivatives would turn out, after lengthy calculation, to be identically zero. Here, though, we may write Ω^2 as a power series in ϵ^2, and compare coefficients. For example, the complete series (which appears to be new) for the square of the ring frequency is

$$\Omega^2 = \frac{4(\alpha-\beta)\beta}{\alpha}\left\{1 + \frac{5\alpha - 8\beta}{12\alpha}\epsilon^2 + \frac{101\alpha^3 - 344\alpha^2\beta + 160\alpha\beta^2 + 128\beta^3}{720\alpha^3}\epsilon^4 + \cdots\right\}. \tag{4.11}$$

In terms of the dimensionless frequency Ω_B, the leading term here is equivalent to $\Omega_B = \sqrt{12}$, as we found after (4.2). The frequency obtained from the first two terms in (4.11) is equivalent to that obtained from the adjusted dispersion relation introduced at the end of §4b, evaluated at $K = 0$; hence in figure 3 it is the intersection of the adjusted second branch with the frequency axis. On the scale of the graph, this frequency is indistinguishable from the exact ring frequency, even

though ϵ is as high as $\frac{1}{2}$. Thus, the first two terms of (4.11) provide a worthwhile approximation to the ring frequency, even for rather thick layers.

Another calculation from (4.5) is the complete series for Ω as a function of K in the neighbourhood of the origin, i.e. on the lowest part of the slow-wave branch. This corresponds to the longest wavelength and lowest frequency which the layer will support. A suitable ansatz is

$$\Omega^2 = \epsilon^2 K^2 \{A_0(\epsilon^2) + A_2(\epsilon^2)K^2 + A_4(\epsilon^2)K^4 + \cdots\}. \tag{4.12}$$

This may be substituted into (4.5), with the coefficients a_0, a_2, b_0, b_2, ... written as series in K with coefficients which are functions of ϵ. On comparing coefficients of successive powers of K, expressions for A_0, A_2, ... are obtained as series in ϵ. The first two of these are

$$A_0 = \frac{(\alpha-\beta)\beta}{3\alpha}\left\{1 - \frac{17}{60}\epsilon^2 + \frac{19}{280}\epsilon^4 + \cdots\right\} \tag{4.13}$$

and

$$A_2 = -\frac{(\alpha-\beta)\beta}{\alpha}\left\{1 - \frac{26\alpha-5\beta}{45\alpha}\epsilon^2 + \frac{3127\alpha^2 - 812\alpha\beta + 168\beta^2}{1512\alpha^2}\epsilon^4 + \cdots\right\}. \tag{4.14}$$

Expressed in terms of Ω_B, the leading terms in (4.12) and (4.13) give the tangent to the dispersion relation at the origin in the form $\Omega_B = \epsilon K$, as we found in §4a when discussing the slow wave. Equations (4.12)–(4.14) determine the dependence of phase speed on the curvature of the layer for the longest low-frequency waves.

(d) Expansions in powers of the thickness

An alternative approach to that of the previous subsection is to start with a series expansion in powers of the dimensionless thickness ϵ for arbitrary wavenumber. Thus for the slow wave, instead of (4.12) one starts with the ansatz

$$\Omega^2 = \epsilon^2 \{\hat{A}_0(K^2) + \hat{A}_2(K^2)\epsilon^2 + \hat{A}_4(K^2)\epsilon^4 + \cdots\}. \tag{4.15}$$

This may be substituted into (4.5), with a_0, a_2, b_0, b_2, ... as given by (4.6)–(4.10). On comparing coefficients of successive powers of ϵ, expressions for $\hat{A}_0$, $\hat{A}_2$, ... are obtained without approximation. The first three of these are

$$\hat{A}_0 = \frac{(\alpha-\beta)\beta}{3\alpha}\frac{K^2(1-K^2)^2}{1+K^2}, \tag{4.16}$$

$$\hat{A}_2 = -\frac{(\alpha-\beta)\beta}{180\alpha^2}\frac{K^2(1-K^2)^2}{(1+K^2)^3}\{\alpha\hat{p}_2(K^2) + \beta\hat{q}_2(K^2)\} \tag{4.17}$$

and

$$\hat{A}_4 = \frac{(\alpha-\beta)\beta}{15\,120\alpha^3}\frac{K^2(1-K^2)^2}{(1+K^2)^5}\{\alpha^2\hat{p}_4(K^2) + \alpha\beta\hat{q}_4(K^2) + \beta^2\hat{r}_4(K^2)\}, \tag{4.18}$$

where

$$\hat{p}_2(K^2) = 17 - 19K^2 + 11K^4 + 27K^6, \quad \hat{q}_2(K^2) = 20K^2(1-K^2)(1+K^2), \tag{4.19}$$

$$\hat{p}_4(K^2) = 342 - 733K^2 + 2754K^4 + 1162K^6 - 1558K^8 + 1171K^{10} + 1182K^{12}, \tag{4.20}$$

$$\hat{q}_4(K^2) = 28K^2(1-K^2)(1+K^2)(29 - 83K^2 + 7K^4 + 59K^6) \tag{4.21}$$

and

$$\hat{r}_4(K^2) = 28K^2(1-K^2)^2(1+K^2)^2(-6 + 19K^2). \tag{4.22}$$

These exact expressions may be compared with formulae in the literature obtained by other methods. For example, equation (3.6) in [17] is equivalent to (4.16) and (4.17) above, but with different expressions for $\hat{p}_2$ and $\hat{q}_2$; and the approach of [18], based on a iterative method, also leads to expansions in the thickness.

The corresponding ansatz for the fast wave is

$$\Omega^2 = \hat{B}_0(K^2) + \hat{B}_2(K^2)\epsilon^2 + \hat{B}_4(K^2)\epsilon^4 + \cdots. \tag{4.23}$$

A similar analysis to the above gives

$$\hat{B}_0 = \frac{4(\alpha-\beta)\beta}{\alpha}(1+K^2) \tag{4.24}$$

and

$$\hat{B}_2 = \frac{(\alpha-\beta)\beta}{3\alpha^3}\frac{1}{1+K^2}\{\alpha^2\tilde{p}_2(K^2) + \alpha\beta\tilde{q}_2(K^2) + \beta^2\tilde{r}_2(K^2)\}, \tag{4.25}$$

where

$$\tilde{p}_2(K^2) = 5 - 7K^2 - 9K^4 - K^6 \tag{4.26}$$

and

$$\tilde{q}_2(K^2) = -4(1-K^2)(1+K^2)(2+K^2), \quad \tilde{r}_2(K^2) = -4K^2(1+K^2)^2. \tag{4.27}$$

Special cases of the formulae obtained in this section, e.g. for $K=0$, agree with those found earlier. Another check with earlier formulae is that, in (4.15), expansion of $\hat{A}_0, \hat{A}_2, \ldots$ in powers of K gives coefficients of successive terms $\epsilon^n K^m$ which agree with these coefficients as determined by (4.12)–(4.14); and similarly for $\hat{B}_0, \hat{B}_2, \ldots$.

5. Theory for a thick layer

(a) Functional form of the dispersion relation

Although we have performed expansions in powers of $\epsilon = h/a$, this is not essential, as closed-form expressions for a_0, a_2, b_0, $b_2, \ldots$ in (4.5) can be found. The calculation is most easily performed in terms of the quantity δ defined by $\mathrm{e}^\delta = \mathrm{a_e}/\mathrm{a_i}$, for which

$$\delta = \ln\left(\frac{1+\epsilon/2}{1-\epsilon/2}\right) = \epsilon + \frac{1}{12}\epsilon^3 + \frac{1}{80}\epsilon^5 + \cdots \tag{5.1}$$

and

$$\epsilon = 2\tanh\tfrac{1}{2}\delta = \delta - \tfrac{1}{12}\delta^3 + \tfrac{1}{120}\delta^5 + \cdots. \tag{5.2}$$

Dimensionless lengths will now be based not on the arithmetic mean $a = (a_\mathrm{i} + a_\mathrm{e})/2$ of the inner and outer radii, but on their geometric mean $\tilde{a} = (a_\mathrm{i}a_\mathrm{e})^{1/2}$, so that $a_\mathrm{i}/\tilde{a} = \mathrm{e}^{-\delta/2}$ and $a_\mathrm{e}/\tilde{a} = \mathrm{e}^{\delta/2}$. The ratio of the two means is $a/\tilde{a} = \cosh\frac{1}{2}\delta$. Dimensionless frequencies corresponding to $\tilde{a}$ are $\tilde{\Omega} = \omega\tilde{a}/c_0$ and $\tilde{\Omega}_\mathrm{B} = \omega\tilde{a}/c_\mathrm{B}$.

The reason for using δ is that it gives expressions in $\mathrm{e}^{\pm\delta}$ and $\mathrm{e}^{\pm 2K\delta}$ rather than $\{(1+\epsilon/2)/(1-\epsilon/2)\}^{\pm 1}$ and $\{(1+\epsilon/2)/(1-\epsilon/2)\}^{\pm 2K}$. Instead of (4.5) we write the exact dispersion relation in the form

$$r_0(K^2,\delta^2) + r_2(K^2,\delta^2)\tilde{\Omega}^2 + r_4(K^2,\delta^2)\tilde{\Omega}^4 + \cdots = 0. \tag{5.3}$$

The functional form of the coefficients $r_0, r_2, \ldots$ is

$$r_n(K^2,\delta^2) = s_n(K^2,\delta) + s_n(K^2,-\delta), \tag{5.4}$$

with s_n given in terms of functions p_n and q_n by

$$s_n(K^2,\delta) = p_n(K,\mathrm{e}^\delta)\mathrm{e}^{2K\delta} + p_n(-K,\mathrm{e}^\delta)\mathrm{e}^{-2K\delta} + q_n(K^2,\mathrm{e}^\delta). \tag{5.5}$$

Thus r_n is even in K and δ, and s_n is even in K; but s_n is not even in δ, and p_n is not even in K. There are several equivalent ways of writing the functions r_n, for example involving $\sinh(2K\delta)$ and $\cosh(2K\delta)$, or involving even and odd functions of K separately instead of using p_n. The

advantage of (5.4) and (5.5) is that, for each n, only the two basic functions p_n and q_n need to be calculated. These have simple expressions in powers of e^{δ}, as indicated by the scheme

$$p_0 = p_{00}, \qquad q_0 = q_{00} + q_{02}e^{2\delta}, \tag{5.6}$$

$$p_2 = p_{21}e^{\delta}, \qquad q_2 = q_{21}e^{\delta} + q_{23}e^{3\delta}, \tag{5.7}$$

$$p_4 = p_{40} + p_{42}e^{2\delta}, \qquad q_4 = q_{40} + q_{42}e^{2\delta} + q_{44}e^{4\delta} \tag{5.8}$$

and

$$p_6 = p_{61}e^{\delta} + p_{63}e^{3\delta}, \qquad q_6 = q_{61}e^{\delta} + q_{63}e^{3\delta} + q_{65}e^{5\delta}, \tag{5.9}$$

and so on. Thus p_n is a polynomial of degree $\frac{1}{2}n$ in e^{δ}, and likewise q_n is of degree $\frac{1}{2}n+2$; these polynomials alternately contain only even powers of e^{δ} or only odd powers, with coefficients p_{nm} and q_{nm} which are functions of K. Hence the only work involved in determining (5.3) is the calculation of a sequence of functions of K, namely the coefficients p_{00}, q_{00}, $q_{02}, \ldots$. These are rational functions, in which the denominators are products of terms $s^2 - K^2$ for $s = 2, 3, \ldots, \frac{1}{2}n + 1$. The coefficients p_{n0} and all q_{nm} are functions of K^2 only; the remaining coefficients, namely p_{nm} for $m \neq 0$, contain both even and odd powers of K in their numerators. A short Mathematica code has no difficulty calculating these functions of K for n up to about $n = 20$; the first two sets of these are

$$\begin{pmatrix} p_{00} \\ q_{00} \\ q_{02} \end{pmatrix} = \frac{4(\alpha-\beta)^2}{\alpha^2\beta^2} K^2(1-K^2)^3 \begin{pmatrix} \frac{1}{2} \\ -(1-K^2) \\ -K^2 \end{pmatrix} \tag{5.10}$$

and

$$\begin{pmatrix} p_{21} \\ q_{21} \\ q_{23} \end{pmatrix} = \frac{\alpha-\beta}{\alpha^3\beta^3} \frac{K^2(1-K^2)^2}{4-K^2} \begin{pmatrix} (2-K)\{\alpha^2(1+4K+K^2) - 2\alpha\beta(2+K) + \beta^2(1-K^2)\} \\ (4-K^2)(\alpha-\beta)\{-\alpha(3-K^2) + \beta^2(1-K^2)\} \\ \alpha^2(8-3K^2+K^4) + 2\alpha\beta K^2(4-K^2) - \beta^2K^2(1-K^2) \end{pmatrix}. \tag{5.11}$$

(b) Absence of singularities

Despite the form of their denominators, the functions r_n are always finite, even at integer values of K, because the numerators are then also zero. For example, in r_0 the numerator of $s_0(K, \pm\delta)$ contains a term $\sinh^2 K\delta - K^2 \sinh^2 \delta$, which is divisible by $\delta^4 K^2(1-K^2)$ in the sense that the function

$$\frac{\sinh^2 K\delta - K^2 \sinh^2 \delta}{\delta^4 K^2(1-K^2)} \tag{5.12}$$

is tame, i.e. analytic, in the neighbourhood of $(K, \delta) = (0, 0)$. This may be seen either by L'Hôpital's rule, or by expanding the numerator as a power series in δ, and noting that every coefficient is divisible by the numerator, to give the series expansion in the form

$$-\tfrac{1}{3}\{1 + \tfrac{2}{15}(1+K^2)\delta^2 + \tfrac{1}{105}(1+K^2+K^4)\delta^4 + \cdots\}. \tag{5.13}$$

The later coefficients $r_2, r_4, \ldots$ in (5.3) are similar, but with longer expressions in the numerators, and with products $(4-K^2)(9-K^2)(16-K^2)\ldots$ in the denominators. On expanding (5.3)–(5.5) in this way, and cancelling a common factor, the result is

$$\tilde{a}_0\delta^2K^2(1-K^2)^2 + \tilde{a}_2\tilde{\Omega}^2 + (\tilde{b}_0 + \tilde{b}_2\delta^2\tilde{\Omega}^2 + \tilde{b}_4\delta^4\tilde{\Omega}^4 + \cdots)\tilde{\Omega}^4 = 0, \tag{5.14}$$

where

$$\tilde{a}_0 = \frac{4(\alpha-\beta)^2}{3\alpha^2}\left\{1+\frac{2}{15}(1+K^2)\delta^2+\frac{1}{105}(1+K^2+K^4)\delta^4+\cdots\right\}, \tag{5.15}$$

$$\tilde{a}_2 = -\frac{4(\alpha-\beta)}{\alpha\beta}\left\{1+K^2+\frac{1}{6\alpha}[\alpha(5+K^2+4K^4)+2\beta K^2(1-K^2)]\delta^2+\cdots\right\}, \tag{5.16}$$

$$\tilde{b}_0 = \frac{1}{\beta^2}\left\{1+\frac{1}{3\alpha^2}[4\alpha^2(1+K^2)+2\alpha\beta(1-K^2)-\beta^2(2+K^2)]\delta^2+\cdots\right\}, \tag{5.17}$$

$$\tilde{b}_2 = -\frac{1}{6\alpha\beta^3}\left\{\alpha+\beta+\frac{1}{30\alpha^2}[3\alpha^3(7+6K^2)+3\alpha^2\beta(19+6K^2) - 2\alpha\beta^2(1+9K^2)-2\beta^3(3+K^2)]\delta^2+\cdots\right\} \tag{5.18}$$

and

$$\tilde{b}_4 = \frac{1}{360\alpha^2\beta^4}\{(\alpha+3\beta)(3\alpha+\beta)+\cdots\}. \tag{5.19}$$

This series expansion may also be obtained from (4.5)–(4.10) by the change of variables (5.2) and $\Omega = \tilde{\Omega}\cosh\frac{1}{2}\delta$.

As $r_0, r_2\ldots$ can be written as closed-form expressions without singularities, higher truncations of the series (5.3) may be plotted without further approximation, corresponding to an expansion of the exact dispersion relation (3.7) in powers of frequency,. This has been done in figure 2 for $\epsilon = \frac{1}{2}$ up to the term in $\tilde{\Omega}^{12}$. The approach of the truncations to the exact curves of the dispersion relation is regular, progressively extending the accuracy higher up any given branch, and also capturing new branches at steady rate. The cut-on frequencies are captured accurately. No numerical problems arise in plotting the functions in this paper at normal resolution using standard IEEE arithmetic; though L'Hôpital's rule or a numerical limit should be used if K is exactly integral.

The functions $r_0, r_2, \ldots$ and their derivatives with respect to K at $K=0$ determine the cut-on frequencies of the different branches, and these may be expanded in powers of δ. For example, the square of the ring frequency, i.e. the first cut-on frequency, is given by

$$\tilde{\Omega}^2 = \frac{4(\alpha-\beta)\beta}{\alpha}\left\{1+\frac{\alpha-4\beta}{6\alpha}\delta^2+\frac{3\alpha^3-72\alpha^2\beta+80\alpha\beta^2+64\beta^3}{360\alpha^3}\delta^4+\cdots\right\}, \tag{5.20}$$

which is equivalent to (4.11).

(c) Neighbourhood of the origin

In the neighbourhood of the origin of the $(\tilde{\Omega}, K)$ plane, we may write

$$\tilde{\Omega}^2 = \delta^2K^2\{\tilde{A}_0(\delta^2)+\tilde{A}_2(\delta^2)K^2+\tilde{A}_4(\delta^2)K^4+\cdots\}. \tag{5.21}$$

Substitution into (5.3), followed by comparison of powers of K, gives $\tilde{A}_0, \tilde{A}_2, \ldots$ in terms of r_0, $r_2, \ldots$ without approximation, for example

$$\tilde{A}_0 = \frac{(\alpha-\beta)\beta}{\alpha}\frac{1}{\delta^2}\left\{1-\frac{\delta^2}{\sinh^2\delta}\right\}\frac{1}{\cosh\delta}. \tag{5.22}$$

This provides an exact expression for the phase speed at the origin, which in dimensionless form based on $\tilde{\Omega}$ and K is $\delta\tilde{A}_0^{1/2}$. The relation $\Omega = \tilde{\Omega}\cosh\frac{1}{2}\delta$ shows that the transformation rule between $\tilde{A}_n$ and the functions A_n defined by (4.12) is $A_n = (\delta^2/\epsilon^2)(\cosh^2\frac{1}{2}\delta)\tilde{A}_n$, so that (5.22) gives, via the relation $\epsilon = 2\tanh\frac{1}{2}\delta$, an exact expression for A_0 in terms of ϵ. When this expression is expanded in powers of ϵ, the result is (4.13), i.e. the expansion for A_0 obtained by proceeding from the outset with series in ϵ. This checks of the accuracy of the calculations. Alternatively,

expansion of (5.22) in powers of δ gives

$$\tilde{A}_0 = \frac{(\alpha-\beta)\beta}{3\alpha}\left\{1 - \frac{7}{10}\delta^2 + \frac{857}{2520}\delta^4 + \cdots\right\}. \tag{5.23}$$

Similarly,

$$\tilde{A}_2 = -\frac{(\alpha-\beta)\beta}{\alpha}\left\{1 - \frac{179\alpha - 20\beta}{180\alpha}\delta^2 + \frac{1230\alpha^2 - 224\alpha\beta + 21\beta^2}{1890\alpha^2}\delta^4 + \cdots\right\}, \tag{5.24}$$

which is equivalent to (4.14).

6. Mathematical aspects

(a) The deferred approach to the limit

We now show that no singularities in K can occur in any of the formulae in this paper, not merely in the coefficients such as r_n. The key observation is that in going from the determinant (3.1), expressed in terms of f and g, to the determinant (3.7), expressed in terms of F and G, we divided by $\{(\sin\pi K)/(\pi K)\}^2$. In the original determinant, only two columns can be linearly independent when K is an integer, i.e. the corresponding matrix becomes of rank 2, because the right-hand sides of (2.16)–(2.17) for such K then involve only one function (because J_{-K} is then proportional to J_K). Therefore, the original determinant contains factors K^2, $(K\pm1)^2$, $(K\pm2)^2, \ldots$, in the sense that when divided by any of these terms it remains finite for all K. The infinite-product expansion of the sine function shows that division by $\{(\sin\pi K)/(\pi K)\}^2$ is equivalent to division by $(K\pm1)^2(K\pm2)^2\ldots$, which therefore cannot introduce any singularities in the transition from (3.1) to (3.7). This establishes the result. The process adopted, of division by a function adapted to integral values of K, is the deferred approach to the limit on which our method is based.

In avoiding singularities, it is necessary to be systematic in keeping all the terms of a given order. Thus, in an expansion in powers of Ω, for example, the coefficient of any power Ω^n must include all the terms in K multiplying this power, to maintain the required cancellation; and likewise for an expansion in powers of δ. An approach which does not satisfy this condition is to expand each of the terms in the determinant (3.7) individually to the same order in Ω, and then evaluate the determinant without further reduction. A contour plot showing where the determinant is zero in the (frequency, wavenumber) plane then contains disfiguring horizontal lines where the plotting algorithm puts spurious zeros between large positive and negative values on either side of the poles in K. No such lines appear when either all or no terms are retained at a given order.

(b) Factorization of the determinant

The curvature of the layer couples bending and stretching waves. Thus in general, the dispersion relation cannot factorize into the product of two separate dispersion relations. An exception occurs for $K=0$, corresponding to waves with no variation of phase along the layer, i.e. no dependence on the angle θ defined in §2a. In this case, there are two separate families of waves, in which the displacement is mainly radial or mainly circumferential, and it follows that the dispersion relation (3.1) or (3.7) must factorize when $K=0$. The factorization is evident at once in our formulation, because the function $g(\Omega, K)$ defined in (2.23), and $G(\Omega, K)$ defined in (3.5), each contain a factor K, and so are identically zero when $K=0$. The g and G terms in the determinants (3.1) and (3.7) are then zero, so that the dispersion relation becomes the two independent equations

$$f^+_{1\mathrm{i}}f^-_{1\mathrm{e}} - f^-_{1\mathrm{i}}f^+_{1\mathrm{e}} = 0, \quad f^+_{2\mathrm{i}}f^-_{2\mathrm{e}} - f^-_{2\mathrm{i}}f^+_{2\mathrm{e}} = 0, \tag{6.1}$$

or equivalently

$$F^+_{1\mathrm{i}}F^-_{1\mathrm{e}} - F^-_{1\mathrm{i}}F^+_{1\mathrm{e}} = 0, \quad F^+_{2\mathrm{i}}F^-_{2\mathrm{e}} - F^-_{2\mathrm{i}}F^+_{2\mathrm{e}} = 0. \tag{6.2}$$

Here, the functions f and F are evaluated at $K=0$; explicitly,

$$f(\Omega,0)=(\tfrac{1}{2})\Omega_2^2 J_0(\Omega)-\Omega J_1(\Omega) \tag{6.3}$$

and

$$F(\Omega,0)=\sum_{s=0}^{\infty}\frac{1}{2}\Omega_2^2+2s\frac{(-(1/4)\Omega^2)^s}{s!}. \tag{6.4}$$

The notation for subscripts and superscripts is as in (2.24)–(2.25); thus the functions used in (6.1)–(6.2) depend on the interior and exterior radii of the layer, a_i and a_e, and hence on ϵ or δ, through the definitions of the dimensionless frequencies $\Omega_{1\mathrm{i}}$, $\Omega_{2\mathrm{e}}$, ... The solutions of these equations give the cut-on frequencies of the different branches of the dispersion relation, i.e. the points where the branches intersect the frequency axis in figures 2 and 3.

If the left-hand sides of equations (6.2) are expanded in powers of Ω and ϵ, the first equation is found to be exactly divisible by $\epsilon K(1-K^2)\Omega^2$, and the second equation by $\epsilon K(1-K^2)\Omega^4$, consistent with the overall factor of $\epsilon^2K^2(1-K^2)^2\Omega^6$ found in §4c. On cancelling these factors, and writing Ω^2 as a series expansion in ϵ^2 beginning with a term of order one, the first equation gives (4.11), indicating that this equation captures the ring frequency, whereas the second equation gives no series solution of this type, other than the trivial solution $\Omega=0$. The dimensionless frequency variable here is the radius-scaled frequency $\Omega=\omega a/c_0$, which takes order-one values when ω is in the low-frequency range of order c_0/a.

To capture high cut-on frequencies, one should rewrite the equations in terms of the thickness-scaled frequency $\Omega_\mathrm{s}=\omega h/c_0$, which takes order-one values when ω is in the high-frequency range of order c_0/h. This second range, higher than the first by a factor $a/h=\epsilon^{-1}$, is the frequency range of the thickness stretch and shear vibrations of a flat plate of thickness h. Expansions in powers of ϵ with Ω_s maintained as an order-one quantity correspond to small perturbations of flat plate results; this gives a fundamentally different type of behaviour from that described by figure 3, because the strong coupling which occurs at very low frequencies cannot be represented as a small perturbation of any flat plate result. Thus, the decision whether to use Ω or Ω_s corresponds to the decision of which regime to analyse.

The factorization (6.1) or (6.2) applies only when K is exactly zero, and so does not by itself provide series expansions in K, either of the type used in §§4c and 5c, or of the type required to capture thickness stretch and shear vibrations. Nevertheless, the factorization is basic to expansions in powers of K, as the leading order equation is necessarily the product of the left-hand sides of (6.2), divided by the common factor $\epsilon^2K^2(1-K^2)^2\Omega^6$; the first correction term involves the quantities $G_{1\mathrm{i}}^+/K,\ldots,G_{2\mathrm{e}}^-/K$, evaluated at $K=0$, together with higher terms in $F_{1\mathrm{i}}^+,\ldots,F_{2\mathrm{e}}^-$. In performing such expansions, it is useful to know at the outset the common factors which must cancel out.

7. Conclusion and further work

All the results in this paper are derived by mathematical deduction from the governing equations of elastic wave motion, rather than from kinematic hypotheses or ad hoc modelling assumptions. It has been shown that by making maximum use of symmetry, and exploiting the exact cancellation of non-analytic terms, very simple and complete results can be obtained; these include the complete series expansions (4.5) and (5.3) for the dispersion relation in powers of the frequency, for which we have provided analytic formulae for the coefficients. A mathematical theme of the paper is analytic structure, especially in relation to the avoidance of logarithmic terms if these are ultimately removable.

The method adopted, that of a deferred approach to a limit, applies to many problems with cylindrical or spherical geometry in mathematical physics. A natural extension of the results is to full three-dimensional geometry, e.g. wave propagation in an annulus, and to different types of wave, e.g. electromagnetic or fluid-dynamical. In three-dimensional problems, waves with low axial wavenumber are particularly important, and involve a non-trivial mathematical limit, which

the method can address. Another extension is take the functional form (5.5) of the coefficients of the dispersion relation as the basis for a multiple scales analysis, in which the scales are K and $K\delta$. Such an approach separates that aspect of the wave motion which depends on the thickness h of the layer, and that part which depends on its radius of curvature a; it is a standard approach in short-wave asymptotic theory [19,20].

This paper has determined very accurate approximations to the dispersion relation, but the expressions we have given for the field are exact. It possible to calculate, analytically, the corresponding approximations to the field for any value of frequency and wavenumber on the dispersion curve; a determination of the analytic structure of the approximate field shapes is a task for future investigation.

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